On gKc-Spaces

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Abstract

This paper is devoted to introduce new concepts so-called gKc-space, minimal gKc-space and locally gKc-space. Several various theorems about these concepts are proved. Further properties are stated as well as the relationships between these concepts with another types of KC-spaces are investigated.

Keywords: g-closed set, g-compact set, Kc-space.

1. Introduction:

It is known that compact subset of Hausdorff space is closed, this motivates the author [1] to introduce the concept of KC-spaces, and these are the spaces in which every compact subset is closed. In 2011 the authors [2] introduce new concepts namely K(gc) and gK(gc)-spaces. The aim of this paper is to continue the study KC-spaces.

2. Preliminaries:

The basic definitions that needed in this work are recalled. In this work, space X means a topological space (X, τ) on which no separation axioms are assumed, unless explicitly stated. The interior and the closure of any subset A of X will be denoted by Int(A) and cl(A) respectively. The author in [3] introduced the following definitions:

A subset F of a space X is called generalized closed (briefly g-closed) if cl(F) ⊆ O, whenever F ⊆ O and O is open in X. A subset A of a space X is called generalized open (briefly g-open), if and only if F ⊆ Int(A) whenever F is closed and F ⊆ A. Also a space X is called $T_{1/2}$-space if every g-closed set in X is closed. The author in [4] introduced the following definitions: A function $f: X \rightarrow Y$ is said to be g-closed if $f(F)$ is g-closed subset of Y, whenever F is closed subset of X, and is said to be g"-closed if $f(F)$ is closed subset of Y, whenever F is g-closed subset of X, also is said to be g""-closed if $f(F)$ is g-closed subset of Y, whenever F is g-closed subset of X. Also f is said to be g""-continuous if $f^{-1}(F)$ is g-closed (g-open) whenever F is g-closed (g-open) subset of Y. Also the g""-continuous image of g-compact is g-compact [2] and every g-closed subset of g-compact space is g-compact [5]. The author in [2] introduced the following: Let f be a homeomorphism function from a space X into space Y, if M is g-compact set in X, then f(M) is also g- compact. And if f is a homeomorphism function from a space X into space Y, and M is g-closed set in X, then f (M) is also g-closed.

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Finally a space $X$ is said to be $gT_1$ if for every two distinct points $x$ and $y$, there exist two $g$-open sets $U$ and $V$ such that $x \in U$ and $y \notin U$, also $x \notin V$ and $y \in V$. The author in [5] introduced the following definition: A subset $K$ of a space $X$ is said to be generalized compact, (briefly $g$-compact) if for every $g$-open cover of $K$ has a finite subcover. And The author in [6] introduced the following definition: A space $X$ is said to be $K_2$ space if $\text{cl}(A)$ is compact, whenever $A$ is compact set in $X$.

Finally The author in [2] introduced the following definitions: A space $X$ is said to be $K(gc)$ or $(gK(gc))$ - space if every compact or $(g$-compact) set in $X$ is $g$-closed.

A space $X$ is said to be $gK_2$ space if $\text{cl}_{g}(A)$ is compact, whenever $A$ is compact set in $X$.

3. The $Kc$-space:

In this section we introduce a generalization of $Kc$-spaces namely $gKc$-space, also we study the properties and facts about this concept and the relationships between this concept and $gK(gc)$ and $K(gc)$-spaces.

First we introduce the following definition:

Definition 3.1: A space $X$ is said to be $gKc$ if every $g$-compact set in $X$ is closed.

So the concept $gKc$-space is a generalization of $Kc$-space and a strong form of $gK(gc)$-spaces.

Example 3.2: Let $(\mathbb{R}, \tau)$ be the indiscrete topology on the real line $\mathbb{R}$, then it is $gK(gc)$-space but not $gKc$, which implies it is not $Kc$-space. And also its $K(gc)$-space. Since if $A$ is any compact proper subset of $\mathbb{R}$, so the only open set which contain $A$ is $\mathbb{R}$, so $\text{cl}(A) \subset \mathbb{R}$. Also it is not $gKc$-space. Since whenever $A$ be $g$-compact proper subset of $\mathbb{R}$ then it is not necessarily is closed since the only closed sets are $\mathbb{R}$ and $\emptyset$.

Definition 3.3: A space $X$ is said to be $g^*K_2$ if $\text{cl}(A)$ is $g$-compact, whenever $A$ is $g$- compact set in $X$.

Definition 3.4: A space $X$ is said to be $g^{**}K_2$- space if $\text{cl}_{g}(A)$is $g$-compact, whenever $A$ is $g$-compact set in $X$.

Theorem 3.5: Every $gKc$-space is $g^*K_2$- space.

Proof: Let $W$ be $g$-compact set in $gKc$- space $X$. So $W$ is closed, that is, $\text{cl}(W)=W$. Hence $\text{cl}(W)$ is $g$-compact. Therefore $X$ is $g^*K_2$.

Lemma 3.6: Every $gK(gc)$-space is $g^{**}K_2$- space.

Proof: Let $W$ be $g$-compact set in $gK(gc)$- space $X$. So $W$ is $g$-closed, that is, $\text{cl}_{g}(W)=W$. Hence $\text{cl}_{g}(W)$ is $g$-compact. Therefore $X$ is $g^{**}K_2$ space.

Theorem 3.7: The $g$ -continuous function $f$ from $g$- compact space $X$ into $gKc$ – space $Y$ is $g$ -closed function.

Proof: Let $F$ be $g$-closed in $X$, where $X$ is $g$- compact space, then by lemma 3.8 $F$ is $g$-compact but $f$ is $g^{**}$-continuous function, then $f(F)$ is $g$-compact in $Y$ by
 lemma 3.7. But Y is gKc-space, which implies that f(F) is closed in Y. Hence f is g*-closed function.

**Corollary 3.8:** The g**-continuous function f from g-compact space X into gKc-space Y is g**-closed function.

**Theorem 3.9:** Every continuous function from g-compact space X into Kc-space Y is g*-closed function.

**Proof:** Let F be g-compact subset of X, so F is g-compact in X by lemma 3.8. Since every g-compact is compact set, so F is compact subset of X. f is continuous function, then f(F) is compact in Y where Y is Kc-space, which implies that f(F) is closed subset of Y. Hence f is g*-closed function.

The prove of the following corollary is easy, hence is omitted.

**Corollary 3.10:** Every continuous function from g-compact space X into K(gc)-space Y is g**-closed.

**Remark 3.11:** The continuous image of gKc-space may be not gKc-space, as in the following example:

Consider \( I_R : (R, \tau_u) \rightarrow (R, \tau_i) \), where \( I_R \) the identity functions on R, \( (R, \tau_u) \) is the usual topology on R and \( (R, \tau_i) \) is the indiscrete topology on the real line R. \( (R, \tau_u) \) is gKc-space, but \( (R, \tau_i) \) is not gKc.

**Theorem 3.12:** Let f be g**-continuous injective function from X into gKc-space Y, then X is gK(gc).

**Proof:** Let F be any g-compact subset in X, then f(F) is g-compact in Y by lemma 3.7. Since Y is gKc-space which implies that f(F) is closed in Y, so it is g-compact. But f is g**-continuous injective function, so \( f^{-1}(f(F)) \) is g-compact in X. Let \( F = f^{-1}(f(F)) \), therefore F is g-compact in X. Hence X is gK(gc).

**Corollary 3.13:** Let f be g**-continuous injective function from \( T_{1/2} \)-space X into gKc-space Y, then X is gKc.

**Lemma 3.14:** If X is \( T_{1/2} \)-space and compact, then it is g-compact space.

**Proof:** Let X be a compact \( T_{1/2} \)-space, to show that X is g-compact. Let \( \{ W_a \}_{a \in \Omega} \) be g-open cover of X, which is \( T_{1/2} \), so every g-open set is open, and then \( \{ W_a \}_{a \in \Omega} \) be an open cover of X. But X is compact so every open cover reduces to a finite subcover.

**Theorem 3.15:** If X is \( T_{1/2} \)-space, then X is gKc if it is K(gc)-space.

**Proof:** Let M be g-compact subset of X, so it is compact. Since X is K(gc)-space, then M is g-closed in X and since X is \( T_{1/2} \), then M is closed subset of X. Hence X is gKc-space.
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**Theorem 3.16:** If \( X \) is \( T_{1/2} \) gKc, then \( X \) is K(gc)-space.

**Proof:** Let \( M \) be compact subset of \( T_{1/2} \)-space \( X \), so it is g-compact by lemma 3.16. Since \( X \) is gKc-space, then \( M \) is closed in \( X \) so it is g- closed subset of \( X \). Hence \( X \) is K(gc) – space.

**Theorem 3.17:** Let \( Y \) be clopen subspace of \( X \), then if \( W \) is g-open in \( X \), then \( W \cap Y \) is g-open in \( Y \).

**Proof:** If \( W \cap Y = \emptyset \), then \( W \cap Y \) is g-open, since only closed subset of \( \emptyset \) is itself and its subset of \( X \), then \( \emptyset = \emptyset \), since \( \emptyset \in \tau \).

If \( W \cap Y \neq \emptyset \), let \( F \) be closed subset of \( Y \) such that \( F \subseteq W \cap Y \), to prove \( F \subseteq \text{Int}(W \cap Y)_{inY} \). Since \( F \subseteq W \cap Y \) then \( F \subseteq W \) and \( F \subseteq Y \). Because \( F \) is closed subset of \( Y \) and \( Y \) is closed subset in \( X \), then \( F \) is closed subset of \( X \). But \( W \) is g-open subset of \( X \), which implies that \( F \subseteq \text{int}(W)_{inX} \), where \( F \subseteq W \). Since \( Y \) is open set of \( X \) and \( F \subseteq Y \), then \( F \subseteq \text{int}(Y)_{inY} \).

Therefore \( F \subseteq \text{int}(W)_{inX} \cap Y \).

But \( \text{int}(W)_{inY} = \text{int}(W)_{inX} \cap Y \).

So \( F \subseteq \text{int}(W)_{inY} \).

Since \( \text{Int}(Y)_{inY} = \text{Int}(Y)_{inX} \cap Y \).

So \( F \subseteq \text{Int}(Y)_{inY} \).

Therefore \( F \subseteq \text{int}(W)_{inY} \cap \text{int}(Y)_{inY} \).

Which implies that \( F \subseteq \text{int}(W \cap Y)_{inY} \).

Hence \( W \cap Y \) is g-open in \( Y \).

**Theorem 3.18:** If \( W \subseteq Y \subseteq X \). Let \( W \) be g-compact subset of \( Y \) and \( Y \) be clopen, subset of \( X \), then \( W \) is g-compact subset of \( X \).

**Proof:** Let \( W \) be g-compact subset of \( Y \), to show \( W \) is g- compact subset of \( X \).
Let \( \{U_{\alpha}\}_{\alpha \in \Omega} \) be g-open cover of \( W \) in \( X \). \( U'_{\alpha} = U_{\alpha} \cap Y \) is g-open in \( Y \) for each \( \alpha \in \Omega \) by theorem 3.19, that is, \( \{U'_{\alpha}\} \) is g-open cover of \( W \) in \( Y \). But \( W \) is g-compact in \( Y \), then there exist a finite subcover of \( W \) in \( Y \) such that \( W \subseteq \bigcup_{i=1}^{p} U'_{\alpha_i} \), then \( W \subseteq \bigcup_{i=1}^{p} (U_{\alpha_i})_{inX} \cap Y \).

Hence \( W \subseteq \bigcup_{i=1}^{p} (U_{\alpha_i})_{inX} \).

Then \( W \) is g-compact in \( X \).

**Theorem 3.19:** The property of space being gKc is a hereditary property on clopen subspace.

**Proof:** Let \( Y \) be clopen subspace of gKc-space \( X \), and \( A \) be any g-compact subset of \( Y \), then its g-compact in \( X \) by theorem 3.20. But \( X \) is gKc-space, then \( A \) is closed in \( X \). But \( Y \cap A = A \) is closed in \( Y \), then \( A \) is closed in \( Y \). Hence \( Y \) is
Theorem 3.20: The property of space being gKc is a topological property.

Proof: Let \( f : X \to Y \) be a homeomorphism function from a gKc space \( X \) into space \( Y \). Suppose \( F \) is a g-compact set in \( Y \), then \( f^{-1}(F) \) is g-compact in \( X \) by lemma 3.22. But \( X \) is gKc space, then \( f^{-1}(F) \) is closed in \( X \), so \( f(f^{-1}(F)) \) is closed in \( Y \).

But \( F = f(f^{-1}(F)) \), then \( F \) is closed in \( Y \). Hence \( Y \) is gKc-space.

The prove of the following corollary is direct, hence is omitted.

Corollary 3.21: Let \( f : (X, \tau) \to (Y, \tau') \) be a homeomorphism function. Then if \( Y \) is gKc-space, then so is \( X \).

Corollary 3.22: The property of space being gK(gc) is a topological property.

Proof: Let \( f : X \to Y \) be a homeomorphism function from a gK(gc) space \( X \) into space \( Y \). Suppose \( F \) is a g-compact set in \( Y \), then \( f^{-1}(F) \) is g-compact in \( X \) by lemma 3.22. But \( X \) is gK(gc) space, then \( f^{-1}(F) \) is g-closed in \( X \) by lemma 3.25, so \( f(f^{-1}(F)) \) is g-closed in \( Y \).

Since \( F = f(f^{-1}(F)) \), then \( F \) is g-closed in \( Y \). Hence \( Y \) is gK(gc)-space.

Theorem 3.23: Every T2-space is gKc-space.

Proof: Let \( X \) be T2-space and \( W \) be g-compact subset of \( X \), so it is compact which implies that it is closed in \( X \). Hence \( X \) is gKc-space.

Corollary 3.24: Every T2-space is gK(gc)-space.

Theorem 3.25: Every gKc-space is T1-space.

Proof: Let \( x \in X \), since \( \{x\} \) is finite set, then it is g-compact in \( X \). But \( X \) is gKc-space, then \( \{x\} \) is closed.

Corollary 3.26: Every gKc-space is gT1-space.

4. On Minimal gKc-spaces:

In this section we introduce a new concept namely minimal gKc-space.

Definition 4.1: Let \( (X, \tau) \) be gKc-space, then \( (X, \tau) \) is said to be minimal gKc (or simply mgKc-space) if \( (X, \tau') \) is not gKc-space where \( \tau' \subset \tau \).

We will use mgKc-space to denote the minimal gKc-space.

Remark 4.2: Every mgKc-space is gKc, but the converse may be not true in general.

Theorem 4.3: The property of space being mgKc is a topological property.

Proof: Let \( (X, \tau_X) \) be a mgKc-space, and \( f : (X, \tau_X) \to (Y, \tau_Y) \) be homeomorphism, and \( (Y, \tau_Y) \) is gKc-space. To prove that \( Y \) is mgKc-space. Assume \( (Y, \tau_Y) \) is not mgKc-space, then there exist a topology \( \tau'_Y \subset \tau_Y \) such that \( (Y, \tau'_Y) \) is a gKc-space. Define \( \tau_i = \{f^{-1}(V) : V \in \tau'_Y \} \). \( \tau_i \) is a topology on \( X \), \( \tau_i \subset \tau_X \) and \( (X, \tau_i) \) is a gKc space but that contract to the fact \( (X, \tau_X) \) is mgKc-space. Hence \( (Y, \tau_Y) \) is mgKc-space.
Proposition 4.5: Let \((X, \tau_X)\) be \(g\)-compact, \(gKc\)-space and \((Y, \tau_Y)\) be subspace of \(X\). Then \(Y\) is \(g\)-compact if and only if \(Y\) is \(g\)-closed in \(X\).

Proof: Let \((X, \tau_X)\) be \(g\)-compact \(gKc\)-space, and \((Y, \tau_Y)\) be subspace of \(X\). Suppose \(Y\) is \(g\)-compact in \(X\) and because \(X\) is \(gKc\)-space, then \(Y\) is closed in \(X\) so it is \(g\)-closed.

Conversely, suppose \(Y\) is \(g\)-closed in \(X\), and since \(Y\) is \(g\)-compact, then \(Y\) is \(g\)-compact by lemma 3.8.

5. On locally \(gKc\)-space:

In this section we introduce a generalization of \(gKc\)-space namely locally \(gKc\)-spaces. We study the relationships between this generalization and \(gKc\)-spaces.

Definition 5.1: A space \(X\) is said to be locally \(gKc\)-space if each point in \(X\) has \(gKc\) neighborhood. So every \(gKc\)-space is locally \(gKc\), but the converse is not true in general. In the next theorem we give the sufficient condition to make the converse is true.

Theorem 5.2: A space \(X\) is a \(gKc\)-space if and only if each point has closed neighborhood which is a \(gKc\)-space.

Proof: If \(X\) is \(gKc\)-space, then for each \(x \in X\), \(X\) itself is a closed neighborhood that is \(gKc\).

Conversely, let \(L\) be \(g\)-compact in \(X\) such that \(x \in X\) and \(x \notin L\).

Choose a closed neighborhood \(W_x\) of \(x\) in \(X\) such that \(W_x\) is \(gKc\)-subspace in \(X\). Then \(W_x \cap L\) is closed in \(L\) and since \(L\) is \(g\)-compact, so \(W_x \cap L\) is \(g\)-compact but \(W_x\) is \(gKc\)-space which implies that \(W_x \cap L\) is closed in \(W_x\).

Because \(W_x\) is closed in \(X\), then \(W_x \cap L\) is closed in \(X\).

\(W_x - (W_x \cap L) = W_x - L\), is a neighborhood of \(x\) disjoint from \(L\). Hence \(L\) is closed in \(X\). Therefore \(X\) is a \(gKc\)-space.

Theorem 5.3: If a space \(X\) has the property that each point in \(X\) has open \(gKc\)-neighborhood, then \(X\) is \(T_1\)-space.

Proof: Suppose \(X\) is not \(T_1\), that is, there exist two distinct points \(x\) and \(y\) in \(X\) such that for every open set \(U\) contain \(y\) also contain \(x\). Since \(y \in X\), so there exists open \(gKc\)-neighborhood \(U\) of \(y\) so \((U, \tau_U)\) is \(T_1\)-space by theorem 3.29. Thus \(\{y\}\) is closed in \(U\). Then \(U - \{y\}\) is open in \(U\), but \(U\) is open in \(X\), then \(U - \{y\}\) is open in \(X\), but \(y \notin U - \{y\}\) and \(x \in U - \{y\}\) \(\forall\). Hence \(X\) is \(T_1\)-space.

Theorem 5.4: If a space \(X\) has the property that each point in \(X\) has closed
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neighbourhood which is gKc, then every clopen subspace is locally gKc.

Proof: Let Y be clopen subspace of X, where X has the property which state above, then X is gKc -space by theorem 5.2, so Y is also gKc-space by theorem 3.21. But every gKc-space is locally gKc, therefore Y is locally gKc.

Corollary 5.5: If X is gKc-space, then every clopen subspace of X is locally gKc.

Theorem 5.6: The property of space being locally gKc is a topological property

Proof: Let \( \langle X, \tau_X \rangle \rightarrow \langle Y, \tau_Y \rangle \)

be a homeomorphism function, where X is locally gKc-space. To show that Y is also locally gKc-space. Let \( y \in Y \), then there exists \( x \in X \) such that \( f(x) = y \), also there exists a gKc neighborhood N of x.

Since f is homeomorphism then f(N) is also gKc neighborhood of y. Hence Y is also locally gKc-space.

Proposition 5.7: A regular locally gKc-space is gKc.

Proof: Suppose X is a regular and locally gKc-space, then every point has a closed neighborhood which is gKc. Then by theorem 5.2, X is gKc.

Proposition 5.8:

If X is a topological group, then X is gKc-space if and only if X is locally gKc – space.

References: