Intuitionistic fuzzy projective geometry

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Abstract: In this paper, we introduce a new model of intuitionistic fuzzy projective geometry. In this model, points and lines play a similar role, like they do in classical projective plane. Furthermore, we will show that this new intuitionistic fuzzy projective plane is closely related to the fibred projective geometry.

Keywords: Intuitionistic, fuzzy, geometry

Introduction:
We introduced a first model of intuitionistic fuzzy projective geometry, this provides a link between the intuitionistic fuzzy versions of classical theories that are very closely related. In another intuitionistic fuzzy model of projective geometries was constructed: fibred projective planes. In this model the role of points and lines is equivalent (this is not the case in the first model), as in the classical case. Points and lines of the base geometry mostly have multiple degrees of membership.

This paper introduces a third model. We first define an intuitionistic fuzzy projective plane in which points and lines play the same role, and such that every point and every line in the base plane possess degree of membership and degree of non-membership. Also we give a definition for an $n$-dimensional intuitionistic fuzzy projective space and we also investigate the link between fibred and intuitionistic fuzzy projective geometry.

Definition 1.1 [1]
A projective plane $\mathcal{P}$ is an incidence structure $(P, B, I)$ with $P$ a set of points, $B$ a set of lines and $I$ an incidence relation, such that the following axioms are satisfied:
A(a) every pair of distinct points are incident with a unique common line.
A(b) every pair of distinct lines are incident with a unique common point.
A(c) $\mathcal{P}$ contains a set of four points with the property that no three of them are incident
With a common line.
A closed configuration $\mathcal{D}$ of $\mathcal{P}$ is a subset of $P \cup B$ that is closed under taking intersection points of any pair of lines in $\mathcal{D}$ and lines spanned by any pair of distinct points of $\mathcal{D}$. We denoted the line in $\mathcal{P}$ spanned by the points $a$ and $b$ by $[a, b]$.

Definition 1.2 [2]
A projective space $\mathcal{H}$ is an incidence structure $(P, B, I)$ with $P$ a set of points, $B$ a set of lines and $I$ an incidence relation, such that the following axioms are satisfied:
A(a) every line is incident with at least two points.
A(b) every pair of distinct points are incident with a unique common line.
A(c) given distinct points $a, b, c, d, e$ such that $[a, b] = [a, c] \neq [a, d] = [a, e]$, there is a point $x \in [b, d] \cap [c, e]$ (Pasach s axiom).

Definition 1.3 [5]
Let $X$ be a nonempty set. An intuitionistic fuzzy set $Z$ on $X$ is an object having the form
Consider the projective plane \( \mathcal{P} = (P, B, I) \). Suppose \( a \in P \) and \( \alpha, \beta \in [0, 1] \). The IF-point \((a, \alpha, \beta)\) is the following intuitionistic fuzzy set on the point set \( P \) of \( \mathcal{P} \):
\[
(a, \alpha, \beta) : P \rightarrow [0, 1]
\]
\[
   a \mapsto \alpha, \quad a \mapsto \beta
\]
for each \( a \in P \setminus \{a\} \).

The point \( a \) is called the base point of the IF-point \((a, \alpha, \beta)\).

An IF-configuration \( (L, \alpha, \beta) \) with base line \( L \) is defined in a similar way.

**Definition 1.6** [3]

The IF-lines \((L, \alpha, \beta)\) and \((M, \sigma, \omega)\) intersect in the unique IF-point \((L \cap M, \alpha \wedge \sigma, \beta \vee \omega)\).

The IF-points \((a, \alpha, \beta)\) and \((b, \sigma, \omega)\) span the unique IF-line \(\langle (a, b), \alpha \wedge \sigma, \beta \vee \omega \rangle\).

**Definition 1.7** [3]

A fibred projective plane \( fP \) on the projective plane \( \mathcal{P} \) consists of a set \( fP \) of IF-points and a set \( fB \) of IF-lines, such that every point and line of \( \mathcal{P} \) is base point and base line of at least one IF-point and IF-line respectively, and such that \((fP, fB)\) satisfies the following intuitionistic fuzzified axioms of a projective plane:

F(a) every pair of IF-points with distinct base points span a unique IF-line.
F(a) every pair of IF-lines with distinct base lines intersect in a unique IF-point.

The projective plane \( \mathcal{P} \) is called the base geometry of \( fP \).

We can construct a fibred projective plane in the following way. Let \( P' \subseteq P \) and \( B' \subseteq B \) be such that the unique closed configuration containing \( P' \cup B' \) is \( P \cup B \). For each element \( x \) of \( P' \cup B' \), we choose arbitrarily a nonempty subset \( \Sigma_x \) of \([0, 1]\) of which the elements are called the initial value of \( x \), and we define a fibred projective plane \( fP \) as follows.

For each \( x \in P' \cup B' \) and for each \( \alpha, \beta \in \Sigma_x \), the element \((a, \alpha, \beta)\) belongs
to $\mathcal{F}P$. This is step 1 of the construction. We now describe another step.

For any pair of IF-points that we already obtained, the IF-line spanned by it also belongs to $\mathcal{F}P$ by definition. Dually, for any pair of IF-lines, the intersection IF-point belongs to $\mathcal{F}P$. The set of all IF-points and of all IF-lines constructed this way in finite number of steps is readily verified to constitute a fibred projective plane. It is clear that every fibred projective plane can be constructed as above. Indeed, one can always take for each element all its corresponding values as initial values.

Now suppose \( \sum_s \) is a singleton for every \( x \in P \cup B \). If \( P' = P \) and \( B' = \phi \), then we call the fibred projective plane mono-point-generated. If \( P' = P \) and \( B' = B \) then the fibred projective plane is called mono-generated. We will restrict ourselves to these two kinds of fibred projective planes.

We see that $\mathcal{F}P$ can be considered as an ordinary projective plane (its base plane $P$) where to every point and line, a set of values from [0,1] are assigned. Also the intuitionistic fuzzy projective plane in definition 1.4 can be considered as an ordinary projective plane, where to every point (and only to points) one (and only one) degree of membership and nonmembership are assigned.

**Example:**

Consider the classical projective plane \( f = GF(2,2) \), the Fano plane. We will construct a mono-point-generated fibred projective plane with base plane \( f \).

We label the 7 points of \( f \) as \( \{a, b, c, d, e, f, g\} \) and the lines as \( \{A, B, C, D, E, F, G\} \), such that:

\[ A = \{a, b, c\}, B = \{c, d, e\}, C = \{e, f, a\}, D = \{a, g, d\}, E = \{b, g, e\}, F = \{b, d, f\} \]

In step 1, we construct the IF-points \((a, 0.9, 0.1), (b, 0.8, 0.2), (c, 0.7, 0.2), (d, 0.6, 0.3), (e, 0.3, 0.4), (f, 0.4, 0.4)\) and \((g, 0.5, 0.3)\) on the points of \( P \), thus the initial values yield the following fibred projective plane:

\[
\begin{align*}
\Sigma_p & = \{(0.3, 0.4), (0.4, 0.4), (0.5, 0.3), (0.6, 0.3), (0.9, 0.1)\}, \\
\Sigma_s & = \{(0.3, 0.4), (0.4, 0.4), (0.5, 0.3), (0.6, 0.3), (0.8, 0.2)\}, \\
\Sigma_d & = \{(0.3, 0.4), (0.4, 0.4), (0.5, 0.3), (0.6, 0.3), (0.7, 0.2)\}, \\
\Sigma_e & = \{(0.3, 0.4), (0.4, 0.4), (0.5, 0.3)\}, \\
\Sigma_r & = \{(0.3, 0.4), (0.4, 0.4), (0.5, 0.3)\} \\
\Sigma_f & = \{(0.3, 0.4), (0.4, 0.4), (0.5, 0.3)\} \\
\Sigma_g & = \{(0.3, 0.4), (0.4, 0.4), (0.5, 0.3), (0.6, 0.3)\}
\end{align*}
\]

### Intuitionistic fuzzy projective plane

In this section we introduce a third model of an intuitionistic fuzzy projective geometry. Like in the fibré model, it also assigns values to the lines of the base geometry. As in the reference geometry model, it assigns only one value to every point (and line) of the base geometry.

**Definition 2.1** [4]

Suppose $P$ is a projective plane \((P, B, I)\).

The intuitionistic fuzzy set \(Z = (\lambda, \mu)\) on \(P \cup B\) is an intuitionistic fuzzy projective plane on \(P\) if:

1. \(\lambda(L) \geq \lambda(p) \land \lambda(q)\) and \(\mu(L) \leq \mu(p) \lor \mu(q)\), for all \(p, q, \in P\).
2. \(\lambda(p) \geq \lambda(L) \land \lambda(M)\) and \(\mu(p) \leq \mu(L) \lor \mu(M)\), for all \(L, M \in P\).

**Definition 2.2**

Consider the fibred projective plane $\mathcal{F}P$ with base plane $P$. For every point \(l \in L\), we only keep the highest degree of membership and lower degree of nonmembership. This results in an intuitionistic fuzzy set $\mathcal{F}_L$ on the base projective plane $P$. The largest of these sets is then taken as the representing set of $\mathcal{F}P$.
The fibred projective plane $\mathcal{P}$, thus:
\[ x \mapsto \sup \Sigma, \quad \text{and} \quad x \mapsto \inf \Sigma, \]

**Theorem 2.3**

The cream of a fibred projective plane is an intuitionistic fuzzy projective plane, and every an intuitionistic fuzzy projective plane can be considered as the cream of a fibred projective plane.

This theorem makes sure the new definition makes sense since fibred projective planes exist and intuitionistic fuzzy projective planes will also exist.

**Example**

By the previous theorem, we know that the example in section 1 gives rise to the following intuitionistic fuzzy projective plane $If$ on the Fano plane $f$:
\[
(a,0.9,0.1), (b,0.8,0.2), (c,0.7,0.2),
(d,0.6,0.3), (e,0.5,0.3), (f,0.5,0.3)
\quad \text{and}
(g,0.5,0.3)
\]
and points
\[
(A,0.8,0.2), (B,0.6,0.3), (C,0.5,0.3),
(D,0.6,0.3), (E,0.5,0.3), (F,0.5,0.3)
\quad \text{and}
(G,0.6,0.3)
\]

**Intuitionistic fuzzy projective spaces**

So far we have only considered 2-dimensional fibred and intuitionistic fuzzy projective geometries in the plane case. We can also define $n$-dimensional fibred and and intuitionistic fuzzy projective geometries, with $n$ and an arbitrary finite integer, such that the previous theorem holds in the general case. Consider the $n$-dimensional projective space $\mathcal{D}$. Call $U_i$ the set of all $i$-dimensional subspaces of $\mathcal{D}$, for all $i: 0 \leq i \leq n-1$.

**Definition 3.1**

Suppose $\mathcal{D}$ is an $n$-dimensional projective space, and $i \leq n$. An IF-subspace $(V_i, \alpha, \beta)$ of dimension $i$ is the following intuitionistic fuzzy set on the set $U_i$ of $\mathcal{D}$:
\[
(V_i, \alpha, \beta): U_i \to [0,1]
\]
\[ V_i \to \alpha \quad \text{and} \quad V_i \to \beta \]
\[ x \to 0 \quad \text{if} \quad x \in U_i \setminus \{V_i\} \]
The subspace $V_i$ is the base subspace of $(V_i, \alpha, \beta)$.

**Definition 3.2**

An $n$-dimensional fibred projective space $\mathcal{D}$ on the $n$-dimensional projective space $\mathcal{D}$ consist of $n$ sets of IF-object : IF-subspaces of dimension $i$, for $0 \leq i \leq n-1$.

Every subspace (of dimension $i$) of $\mathcal{D}$ is base subspace of at least one IF-subspace (of dimension $i$). Moreover the following axioms have to be fulfilled:

- **F1** the intersection of two IF-subspaces (with distinct base subspaces that are not disjoint) is again an IF-subspace.
- **F2** every two IF-subspaces (with distinct base subspaces that do not span $\mathcal{D}$ itself) span an IF-subspace.

For $i = 0,1,2, n-1$, the IF-subspaces of dimension $i$ will be called IF-points, IF-lines, IF-planes and IF-hyperplanes.

The cream of an $n$-dimensional fibred projective space is defined in the same way as for a fibred projective plane (see definition 3.1).

**Definition 3.3** [4]

Suppose $\mathcal{D}$ is an $n$-dimensional projective space as defined above. The intuitionistic fuzzy set $I\langle \lambda, \mu \rangle$ on $\bigcup_{i=0}^{n-1} U_i$ is a intuitionistic fuzzy projective space of dimension $n$ on $\mathcal{D}$ if for all subspaces $V_i, V_j, V_k, 0 \leq i, j, k \leq n-1$ we have:

1) $\lambda(V_i) \geq \lambda(V_j) \wedge \lambda(V_k)$ and $\mu(V_j) \leq \mu(V_i) \vee \mu(V_k), \forall V_j$ and $V_k$, $V_j \neq V_k$ such that $V_j \cap V_i = V_i \neq \emptyset$.

2) $\lambda(V_i) \geq \lambda(V_j) \wedge \lambda(V_k)$ and $\mu(V_j) \leq \mu(V_i) \vee \mu(V_k), \forall V_j$ and $V_k$, $V_j \neq V_k$ such that $\langle V_j, V_i \rangle = V_i \neq \emptyset$.

**Theorem 3.4**

The cream of an $n$-dimensional fibred projective space is an $n$-dimensional intuitionistic fuzzy projective space, and every $n$-dimensional
intuitionistic fuzzy projective space can be considered as the cream of an $n$-dimensional a fibred projective space.

References


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