Decomposition of Homeomorphisms on Intuitionistic Topological Spaces.

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Abstract
The aim of this paper is to introduce two new classes of generalized homeomorphisms on intuitionistic topological spaces and shown that one of these classes has a group structure. Moreover some properties of these two homeomorphisms are obtained.

Keywords- Generalized closed set, homeomorphism, gsg-homeomorphism, sgs-homeomorphism on ITS.

INTRODUCTION

Throughout the present paper, \((X,T)\) and \((Y,\Psi)\) denote an intuitionistic topological spaces on which no separation axioms are assumed unless explicitly stated. Let \(A\) be an IS in \(X\) we denote the interior of \(A\) (respectively the closure of \(A\) with respect to an intuitionistic topological space)(ITS) \(T\) by \(\text{int}(A)\) (respectively \(\text{cl}(A)\)).

PRELIMINARIES
Since we shall use the following definitions and some properties, we recall them in this section.

- A subset \(B\) of ITS \((X,T)\) is said to be semi-closed if there exists a closed set \(F\) such that \(\text{int}(F) \subseteq B \subseteq F\). A subset \(B\) of \((X,T)\) is called a semi-open set, its complement \(\overline{B}\) is semi-closed in \((X,T)\). Every closed (respectively open) set is semi-closed (respectively semi-open)[10].
- Let \(X\) be a non-empty set, and let \(A\) and \(B\) are IS, having the form \(A = (x, A_1, A_2)\), \(B = (x, B_1, B_2)\) respectively. Furthermore, let \(\{A_t | t \in I\}\) be an arbitrary family of IS in \(X\), where \(A_t = (x, A_t^{(1)}, A_t^{(2)})\), then,
  1) \(\emptyset = (x, \emptyset, X)\) \(\Rightarrow (x, X, \emptyset)\)
  2) \(A \subseteq B\), iff \(A_1 \subseteq B_1\) and \(A_2 \subseteq B_2\)
  3) the complement of \(A\) is denoted by \(\overline{A}\) and defined by \(\overline{A} = (x, A_2, A_1)\).
Let $X, Y$ be non-empty sets and let $f : X \rightarrow Y$ be a function. 

a) If $B$ is IS in $Y$, then the preimage of $B$ under $f$ is denoted by $f^{-1}(B)$ is IS in $X$ defined by $f^{-1}(B) = \{ x \in X | f(x) \in B \}$.

b) If $A$ is an IS in $X$, then the image of $A$ under $f$ is defined $f(A) = \{ f(a) | a \in A \}$.

A map $f : (X, T) \rightarrow (Y, \Psi)$ is said to be semi-closed if the image of each closed $F$ in $(X, T)$ is semi-closed in $(Y, \Psi)$. Every closed mapping is semi-closed [10,11].

An intuitionistic topology (IT, for short), on a non-empty set $X$, is a family $T$ of an IS in $X$ containing $\emptyset$ and $X$ and closed under arbitrary unions and finitely intersections. The pair $(X, T)$ is called an Intuitionistic topological space (ITS, for short).

Let $(X, T)$ be ITS and $A$ be a subset of $X$. Then, the semi-interior and semi-closure of $A$ are defined by:

$Stt(A) = \bigcup \{ G | G \text{ is semi-closed in } X \text{ and } G \subseteq A \}$

$Scn(A) = \bigcap \{ H | H \text{ is semi-closed in } X \text{ and } A \subseteq H \}$

[9].

A subset $B$ of IT $S(X, T)$ is said to be semi generalized-closed (sg-closed) if $Scl(B) \subseteq U$ whenever $B \subseteq U$ and $U$ is semi-open. The complement of a sg-closed set is called sg-open. Every semi-closed set is sg-closed. The concepts of g-closed sets and sg-closed set are in general, independent. The family of all sg-closed set of any ITS $(X, T)$ is denoted by $sgc(X)$[8].

A subset $B$ of ITS $(X, T)$ is said to be generalized semi-open( gs-open) if $F \subseteq Stt(B)$ whenever $F \subseteq B$ and $F$ is closed. $B$ is generalized semi-closed (gs-closed) if and only if $B$ is gs-open. Every closed set (semi-closed set, g-closed set and sg-closed set) is gs-closed. The family of all gs-closed set of any ITS $(X, T)$ is denoted by $gsc(X)$[8].

A map $f : (X, T) \rightarrow (Y, \Psi)$ is called a semi generalized-continuous (sg-continuous mapping) if $f^{-1}(V)$ is sg-closed in $X$ for every closed set $V$ of $Y$.[8].

A map $f : (X, T) \rightarrow (Y, \Psi)$ is called a generalized semi-continuous (gs-continuous mapping) if $f^{-1}(V)$ is gs-closed in $X$ for every closed set $V$ of $Y$.[8].

A map $f : (X, T) \rightarrow (Y, \Psi)$ is called semi-generalized closed map (respectively semi-generalized open map) if $f(V)$ is semi-closed (respectively semi-closed open) in $Y$ for every closed (respectively, open) set $V$ in $X$. Every semi-closed map is semi-generalized-closed. A semi generalized-closed map (resp. semi generalized-open map) is written as sg-closed map (resp. sg-open map).[8].

A map $f : (X, T) \rightarrow (Y, \Psi)$ is called generalized semi-closed map (respectively generalized semi-open map) if $f(V)$ is gs-closed (respectively gs-open) in $Y$ for every closed (respectively, open) set $V$ in $X$. Every semi-closed map is generalized semi-closed. A generalized semi-closed map (resp. generalized semi-open map) is written as gs-closed map (resp. gs-open map).[8].

A map $f : (X, T) \rightarrow (Y, \Psi)$ is said to be a semi homeomorphism (B) (simply s.h.B) if $f$ is continuous, $f$ is semi-open(i.e. $f(U)$ is semi-open for every open set $U$ of $X$) and $f$ is bijective[11].

A map $f : (X, T) \rightarrow (Y, \Psi)$ is said to be a semi homeomorphism (C.H.) (simply s.h.(C.H)) if $f$ is irresolute(i.e. $f(U)$ is semi-open for every semi-open set $V$ of $Y$), $f$ is pre-semi-open(i.e.$f(U)$ is semi-
open for every semi-open set $U$ of $X)$ and $f$ is bijective [8].

- A map $f: (X, T) \rightarrow (Y, \Psi)$ is said to be $sg$-irresolute map if $f^{-1}(V)$ is $sg$-closed set in for every $sg$-closed set $V$ of $Y$[8].

- A map $f: (X, T) \rightarrow (Y, \Psi)$ is said to be $gs$-irresolute map if $f^{-1}(V)$ is $gs$-closed set in for every $gs$-closed set $V$ of $Y$[8].

- A bijection $f: (X, T) \rightarrow (Y, \Psi)$ is called a semi-generalized homeomorphism (sg-homeomorphism) if $f$ is both $sg$-continuous and $sg$-open [8].

- A bijection $f: (X, T) \rightarrow (Y, \Psi)$ is called a g-homeomorphism if $f$ is $g$-continuous and $g$-closed [8].

- A bijection $f: (X, T) \rightarrow (Y, \Psi)$ is called a gc-homeomorphism if $f$ is $g$-irresolute and its inverse $f^{-1}$ is also $g$-irresolute [8].

- A bijection $f: (X, T) \rightarrow (Y, \Psi)$ is called a sg-homeomorphism if $f$ is $sg$-irresolute and its inverse $f^{-1}$ is also $sg$-irresolute [8].

- A bijection $f: (X, T) \rightarrow (Y, \Psi)$ is called a generalized semi-homeomorphism (gs-homeomorphism) if $f$ is both $gs$-continuous and $gs$-open [8].

- A bijection $f: (X, T) \rightarrow (Y, \Psi)$ is called a gsc-homeomorphism if $f$ is $gs$-irresolute and its inverse $f^{-1}$ is also $gs$-irresolute [8].

- An ITS space $(X, T)$ is called if every $g$-closed set is closed, that is if and only if every $gs$-closed set is semi-closed [4].

- An ITS $(X, T)$ is called a $T_{1/2}$ space if every $gs$-closed set is closed [4].

**Remark 2.1**

- The notions $g$-closed set and sg-closed set in ITS are independent notions. The following examples show the cases.

**Example 2.2** Let $X = \{a, b, c\}, T = \{\emptyset, X, A, B\}$, where $A = (x, [a], [b]), B = (x, [a, b], \emptyset)$.

**Example 2.3** Let $X = \{a, b, c\}, T = \{\emptyset, X, A, B\}$, where $A = (x, [a], [b]), B = (x, [a, b], \emptyset)$.

**Example 3.2** Let $X = \{1, 2, 3\}, T = \{\emptyset, X, A, B, C\}$, where $A = (y, [a], [b], [c])$, $B = (y, [a], [b])$, and $C = (y, [a], [b])$.

**GS HOMEOMORPHISM**

In this section, the relation between semi-homeomorphism (B) and gsc-homeomorphism is investigated and the diagram of implications is given. Also the gsg-homeomorphism is defined and some of its properties are obtained.

**Remark 3.1**

The following two examples show that the concept of

1. Semi-homeomorphism (B) and gsc-homeomorphism are independent of each other.

2. Semi-homeomorphism (C.H) and gsc-homeomorphism are independent.

We start with example showing that:

1. gsc-homeomorphism, but not semi-homeomorphism (B) and not homeomorphism (C.H).

2. gsg-homeomorphism, but not g-homeomorphism and no gsc-homeomorphism.

3. gsg-homeomorphism, but not semi-homeomorphism (B) and not homeomorphism (C.H).

**Example 3.2**

Let $X = \{1, 2, 3\}, T = \{\emptyset, X, A, B, C\}$, where $A = (x, [a], [b], [c])$, $B = (x, [a], [b])$, and $C = (x, [a], [b])$.

Let $Y = \{a, b, c\}, T = \{\emptyset, Y, D, E, F\}$, where $D = (y, [a], [b], [c]), E = (y, [a], [b], [c])$, and $F = (y, [a], [b], [c])$. Define a mapping $f: X \rightarrow Y$ by $f(1) = a, f(2) = b$, and $f(3) = c$.

**SOX** = $T \cup \{H, V\}$, where $V = (x, [1, 2, 3]), H = (x, [1, 2], [c])$.
where $M = \{x, [1], [2], 3\}$, $D = \{x, [2], 4\}$, and $I = \{x, [2], 3\}$. Therefore, we can see that 
(1) $f$ is gsc-homeomorphism, but $f$ is not semi-homeomorphism (B) and not semi-homeomorphism (C.H), and (2) $f$ is gsg-homeomorphism, but not homeomorphism (g-homeomorphism, gc-homeomorphism, semi-homeomorphism (B) and not homeomorphism (C.H)).

The next example shows that there is a semi-homeomorphism (B), but not gsc-homeomorphism.

**Example 3.3**

Let $X = \{a, b, c\}$, $T = \{a, X, A, B\}$, where $A = \{x, \{a\}, \{a, c\}\}$ and $B = \{x, \{b\}, \{c\}\}$. Define a mapping $f : X \rightarrow Y$ by $f(a) = 2, f(b) = 3$ and $f(c) = 1$.

We can see that $f$ is semi-homeomorphism (B).

**Proposition 3.5**

The following implications are valid while the reverse implications are not.

The following result gives the relation among different type of homeomorphism defined above.

**Example 3.6**

Let $X = \{a, b, c\}$, $T = \{a, X, A, B, C\}$, where $A = \{x, \{a\}, \{a, b\}\}$ and $B = \{x, \{a\}, \{a, c\}\}$. Define a mapping $f : X \rightarrow Y$ by $f(a) = 1, f(b) = 2$ and $f(c) = 3$.

We can see that $f$ is semi-homeomorphism (C.H), but not gsc-homeomorphism.

**Proposition 3.5**

The following implications are valid while the reverse implications are not.

The following result gives the relation among different type of homeomorphism defined above.

**Example 3.6**

Let $X = \{y, [1], [2], 3\}$, $T = \{a, X, A, B, C\}$, where $A = \{x, [a], [b]\}$ and $B = \{x, [a], [c]\}$. Define a mapping $f : X \rightarrow Y$ by $f(a) = 1, f(b) = 2$ and $f(c) = 3$.

We can see that $f$ is semi-homeomorphism (C.H), but not gsc-homeomorphism.

**Proposition 3.5**

The following implications are valid while the reverse implications are not.
We can see that:

\( f \) is \( g \)-homeomorphism, but not \( g(s) \)-homeomorphism.

\( f \) is \( g(s) \)-homeomorphism, but not \( g(s)c \)-homeomorphism.

\( f \) is \( s \)-homeomorphism, but not \( sgs \)-homeomorphism.

\( f \) is \( gsc \)-homeomorphism, but not \( sgs \)-homeomorphism.

\( f \) is \( sg \)-homeomorphism, but not \( sgs \)-homeomorphism.

\( f \) is \( gsc \)-homeomorphism, but not \( sgs \)-homeomorphism.

**Example 3.7**

a. Recall Example 3.2, we can get the following:

b. \( f \) is \( g \)-hom, but not \( g \)-hom;

c. \( f \) is \( g \)-homeomorphism, but not \( g \)-homeomorphism.

d. \( f \) is \( s \)-homeomorphism, but not \( s \)-hom.

e. \( f \) is \( s \)-homeomorphism, but not \( s \)-homeomorphism.

**Example 3.8**

Let \( X = \{a, b, c\}; T = \{\emptyset, X, A, B\} \) where \( A = \{\{a, b\}, \{\}\}\) and \( B = \{\{a, c\}, \emptyset\} \). Let \( Y = \{1, 2, 3\}; \Psi = \{\emptyset, Y, C, D\} \) where \( C = \{y, \{2, 3\}\} \) and \( D = \{y, \{1, 3\}\} \). Define a mapping

\( f : X \to Y \) by \( f(a) = 1, f(b) = 2 \) and \( f(c) = 3 \).

we can see: \( f \) is \( g \)-homeomorphism, but not \( g \)-homeomorphism.

**Example 3.9**

Recall Example 3.3. It is clear that \( f \) is \( s \)-homeomorphism (B), but not \( s \)-homeomorphism. Recall Example 3.4, we see that \( f \) is \( s \)-homeomorphism (CH), but \( f \) is not \( s \)-homeomorphism.

**Remark 3.10**

a. \( g \)-homeomorphism and \( g \)-homeomorphism are independent.

b. \( s \)-homeomorphism and \( s \)-homeomorphism are independent.

**Note that Example 3.2 show the case:** \( f \) is \( g \)-homeomorphism, but \( f \) is not \( s \)-homeomorphism (B) and not \( s \)-homeomorphism (CH).

Example 3.7 (b) show that \( f \) is \( g \)-hom, but not \( g \)-hom and not \( g \)-homeomorphism.

**Example 3.11**

Let \( X = \{x, \{b, c\}\}; T = \{\emptyset, X, A, B\} \) where \( A = \{x, \{a\}, \emptyset\} \) and \( B = \{x, \{a, b\}\} \). Let \( Y = \{1, 2, 3\}; \Psi = \{\emptyset, Y, C, D\} \) where \( C = \{y, \{2, 3\}\} \) and \( D = \{y, \{1, 3\}\} \). Define a mapping

\( f : X \to Y \) by \( f(a) = 1, f(b) = 2 \) and \( f(c) = 3 \).

we can see that: 1) \( f \) is \( s \)-homeomorphism (B), but not \( s \)-homeomorphism (CH). 2) \( f \) is \( s \)-homeomorphism, but not \( s \)-homeomorphism. 3) \( f \) is \( g \)-homeomorphism and \( g \)-homeomorphism, but not \( g \)-homeomorphism.

**Remark 3.11**

The concepts of \( s \)-homeomorphism (B) and \( s \)-homeomorphism (CH) are independent.

The following examples show the cases.

**Example 3.13**

We show in this example that \( f \) is \( s \)-homeomorphism (CH), but not \( s \)-homeomorphism (B).

Let \( X = \{a, b, c\}; T = \{\emptyset, X, A, B\} \) where \( A = \{x, \{b\}, \{a, c\}\} \) and \( Y = \{x, \{b\}, \emptyset\} \). Let \( Y = \{1, 2, 3\}; \Psi = \{\emptyset, Y, C, D\} \) where \( D = \{y, \{2, 1\}\} \) and \( E = \{y, \{1, 2\}\} \). Define a mapping

\( f : X \to Y \) by \( f(a) = 1, f(b) = 2 \) and \( f(c) = 3 \).

we can see that \( f \) is \( g \)-homeomorphism (B), but not \( g \)-homeomorphism (CH).

**Example 3.14**

Let \( X = \{a, b, c\}; T = \{\emptyset, X, A, B, C\} \) where \( A = \{x, \{a, b\}, \emptyset\} \) and \( B = \{x, \{a, c\}\} \). Let \( Y = \{1, 2, 3\}; \Psi = \{\emptyset, Y, C, D\} \) where \( D = \{y, \{2, 3\}\} \). Define a mapping

\( f : X \to Y \) by \( f(a) = 1, f(b) = 2 \) and \( f(c) = 3 \).

we can see that \( f \) is \( g \)-homeomorphism (B), but not \( g \)-homeomorphism (CH).
mapping $f: X \to Y$ by $f(a) = 1, f(b) = 2$, and $f(c) = 3$. 
$SIX = \{a, \{a\}\} = \{a\}$, $SIB = \{b, \{b\}\} = \{b\}$, $SIS = \{c, \{c\}\} = \{c\}$. 
$SOY = \{y_1, y_2, y_3\}$, where

$\beta_1 = \{y_1, \{1\}\}, \beta_2 = \{y_2, \{2\}\}, \beta_3 = \{y_3, \{3\}\}$. 

We can see that $f$ is semi-homeomorphism (B). But $f$ is not semi-homeomorphism (C.H). 

Next we are going to define the gsg-irresolute and gsg-homeomorphism. 

**Definition 3.15**

A mapping $f: (X, T) \to (Y, \Psi)$, where $(X, T)$ and $(Y, \Psi)$ are ITS is called gsg-irresolute map if the inverse image of every gs-closed set in $Y$ is sg-closed set in $X$. 

**Definition 3.16**

A bijective mapping $f^*: (X, T) \to (Y, \Psi)$, where $(X, T)$ and $(Y, \Psi)$ are ITS is called gsg-homeomorphism, if both $f$ and its inverse are gsg-irresolute. If there exists a gsg-homeomorphism between $(X, T)$ and $(Y, \Psi)$, then $(X, T)$ and $(Y, \Psi)$ are said to be gsg-homeomorphic. The family of all gsg-homeomorphic ITS to ITS $(X, T)$ is denoted by $gshg(X)$. 

Note: Example 3.11, 3.4 and 3.2 shows that gsg-homeomorphism and gsg-homeomorphism (sh(C.H), sh(B), g-hom. and gc-hom.) are independent.

**Remark 3.17**

Every gsg-homeomorphism implies both gsc-homeomorphism and sgsc-homeomorphism. 

**Proof:** Since $f$ is gsg-homeomorphic, so $f$ is gsc-irresolute. Let $V$ be any gs-closed in $Y$, then $f^{-1}(V)$ is sg-closed in $X$. Since $f$ is gsg-irresolute then $f^{-1}(V)$ is sg-closed for every sg-closed in $Y$. Also since $f$ is gsg-homeomorphism, then $f^{-1}$ is gsg-irresolute (by definition). Let $V$ be gs-closed set in $X$, $(f^{-1})^{-1}(V) = f^{-1}(V)$ is sg-closed in $X$. So $f(V)$ is sg-closed for every sg-closed in $Y$. Thus $f$, $f^{-1}$ is sg-irresolute. Therefore $f$ is gsc-homeomorphism. It is clear that $f$ is gsg-homeomorphism, then $f$ is gsc-homeomorphism.

However the converse is not true as shown by the following example. 

**Example 3.18**

Let $A = \{a, b, c\}, B = \{\phi, \omega, A, B, C\}$, where $A = \{x, \{a, b, c\}\}, B = \{x, \{a, c\}\}, C = \{x, \{\alpha\}\}$. 

Let $Y = \{1, 2, 3\}, \Psi = \{\phi, \beta, D, E, F\}$, where $D = \{y, \{1, 2\}\}, E = \{y, \{1, 3\}\}, \Psi = \{\phi, \beta\}$. We can see that $f$ is both gsc-homeomorphism and gsg-homeomorphism. Since $gsc(X) = \{\phi, \omega, \beta, A\}$, $sgc(X) = \{\phi, \beta, A\}$ and $gsc(Y) = \{\phi, \beta, A\}$, so $f$ is not gsg-homeomorphism. 

The following proposition is a direct consequence of definitions. 

**Proposition 3.19**

Every gsg-homeomorphism implies both gsc-homeomorphism and sgsc-homeomorphism. 

**Example 3.3** shows that the converse of (Prop.3.19) is not true in general.

Next we are going to generalize the definition of cardinal number for IS. 

**Definition 3.20**

Let $X$ be any set and $A = \{A_1, A_2\}$ be IS in $X$, then the cardinal number of $A$ is denoted by $\#A$ defined as follows $\#A = (A_1, A_2)$ where $A_1 = \#A_1, A_2 = \#A_2$. 

**Remark**

From definition 3.20 we can get the following properties of cardinal number of IS. 

1. If $\#A = (\alpha, \beta)$ and $\#B = (\gamma, \delta)$, then
   
   $\#A \neq \#B$ if and only if $\alpha = \gamma$ and $\beta = \delta$. 

2. $\#(A \cup B) = (\alpha \cup \gamma, \beta \cup \delta)$, whenever $A \cap B = \emptyset$. 


Definition 3.21

Let \((X,T)\) and \((Y,\Psi)\) be ITS. If the following properties are satisfied;

a. \(\#A = \alpha (\alpha \in \mathbb{N})\)

Then there exists a bijective map \(\Psi: gsc(X) \rightarrow gsc(Y)\) such that for each \(A \in gsc(X)\), \(\#\Psi(A) = \#(A)\).

The space \((X,T)\) and \((Y,\Psi)\) are called S-related.

The following theorem follows from definition of gsg-homeomorphism and (Def.3.21).

Theorem 3.22

The space \((X,T)\) and \((Y,\Psi)\) are gsg-homeomorphic if and only if these spaces are S-related.

Theorem 3.23

a. Every gsc(sgc) homeomorphism from \(X\) onto itself is a gsg-homeomorphism.

b. Every gs(sg) homeomorphism from \(X\) onto itself is gsg-homeomorphism.

Proof:

Since for any \(T_{1/2}\) the family of sg-closed set is equal to the family of gs-closed sets. Any gsc (sgc) homeomorphism from X onto X is gsg-homeomorphism. In any \(T_{1/2}\) every gs-closed subset is a closed subset so (b) is obvious.

d. As direct consequences of theorem 3.23, we have the following corollary.

Corollary 3.24

Let \((X,T)\) and \((Y,\Psi)\) be any ITS’s. if there exist any gsg-homeomorphism from X to Y, then every gsc(sgc)-homeomorphism from X to Y is sgc(sgc)-homeomorphism.

The following theorem gives relations among gsg, gsc, gs, sgc and sg-homeomorphism.

Theorem 3.25

Let \((X,T)\) be ITS, then the following inclusions holds.

a. \(gsh(X) \subseteq gsch(X) \subseteq gsh(X)\)

b. \(gsg(X) \subseteq sgch(X) \subseteq gsh(X)\)

c. If \(gsg(X) \neq \emptyset\), then \(gsg(X)\) is a group and \(gsch(X) = gsch(X) = gsh(X)\).

Proof:

It follows from Remark 3.17 and Cor.3.24.

Theorem 3.26

If \(f: (X,T) \rightarrow (Y,\Psi)\) is gsg-homeomorphism, then it induces an isomorphism from the group \(gsg(X)\) onto the group \(gsg(Y)\).

Proof:

The homomorphism \(f_*: gsg(X) \rightarrow gsg(Y)\) is induced from \(f\) by \(f_*(h) = f \circ h \circ f^{-1} \forall h \in gsg(X)\). Then it easily follows that \(f_*\) is an isomorphism.

SGS-HOMEOMORPHISM

In this section we introduce a new kind of homeomorphism namely sgs-homeomorphism and related to other kind of homeomorphism which are defined in this work.

Definition 4.1

A map \(f: (X,T) \rightarrow (Y,\Psi)\) is called a sgs-irresolute map if the inverse image of each sg-closed set in Y is gs-closed in X.

Definition 4.2

A bijective map \(f: (X,T) \rightarrow (Y,\Psi)\) is called a sgs-homeomorphism if f and its inverse are both sgs-irresolute maps.

If there exists a sgs-homeomorphism from X to Y, then the spaces \((X,T)\) and \((Y,\Psi)\) are said to be sgs-homeomorphic spaces.

Remark 4.3

Every sgc-homeomorphism and gsc-homeomorphism implies a sgs-homeomorphism. But the converse is not true in general see (Example 3.4), which is shows that sgs-homeomorphism is not gsc-homeomorphism.

The following example shows that sgs-homeomorphism is not sgc-homeomorphism and not sg-homeomorphism.

Example 4.4
Let $X = \{a, b, c\}; Y = \{\phi, \emptyset, A\}$, where $A = \{x, \{c\}, \{b\}\}$, Let $Y = \{1, 2, 3\}; \mathfrak{Y} = \{\emptyset, I, B, C\}$, where $B = \langle y, \{1\}, \{2, 3\}\rangle, C = \langle y, \{2, 3\}, \{1\}\rangle$. Define a mapping

$$f: X \to Y \text{ by } f(a) = 2, f(b) = 1 \text{ and } f(c) = 3.$$ 

We can see that $f$ is $gs$-homeomorphism but not $sgs$-homeomorphism, $sg$-homeomorphism and $f$ is not homeomorphism. Also $f$ is $sgs$-homeomorphism, but $f$ is not $gs$-homeomorphism.

**Remark 4.5**

Every homeomorphism is a $sgs$-homeomorphism.

**Proof:** Since $f$ is homeomorphic. So $f$ is continuous. Let $V$ be any closed set in $Y$, then $f^{-1}(V)$ is $gs$-closed set in $X$. Since $f$ is continuous. So by every closed set is $gs$-closed and $sg$-closed set, then $f^{-1}(V)$ is $gs$-closed for every $sg$-closed in $Y$. Thus $f$ is $sgs$-homeomorphism.

Also $f^{-1}$ is continuous. Let $V$ be a closed set, then $(f^{-1})^{-1}(V) = f(V)$ is $gs$-closed in $X$ for every $sg$-closed in $Y$. Thus $f^{-1}$ is $sgs$-irresolute. Therefore $f$ is $sgs$-homeomorphism. But the converse is not true. See example 4.4.

**Remark 4.6**

Every $sgs$-homeomorphism is a $gs$-homeomorphism.

**Proof:** Since $f$ is $sgs$-homeomorphism. So $f$ is $sgs$-irresolute. Let $V$ be any $sg$-closed set in $Y$, then $f^{-1}(V)$ is $gs$-closed set in $X$. Since $f$ is $sgs$-irresolute. So by every closed set is $sgs$-closed set, then $f^{-1}(V)$ is $gs$-closed in $X$ for every closed in $Y$. Thus $f$ is $gs$-continuous and also $f^{-2}$ is $sgs$-irresolute. Let $V$ be a $sgs$-closed set, then $(f^{-2})^{-1}(V) = f(V)$ is $gs$-closed in $X$ for every closed in $Y$. Thus $f^{-2}$ is $gs$-closed. Therefore $f$ is $gs$-homeomorphism. But the converse is not true. See example 3.6.

**Example 4.7**

The mapping $f: (X, \mathcal{T}) \to (Y, \mathfrak{Y})$ in example 3.6 is $gs$-homeomorphism, but note $sgs$-homeomorphism.

**Result 4.8**

a. From example 4.7 we see that any $sgs$-homeomorphism is not a $sgs$-homeomorphism.

b. Every $sgs$-homeomorphism is $gs$-homeomorphism, but the converse is not true. Example 4.4 shows the case.

**Theorem 4.9**

a. Every $sgs$-homeomorphism from a $T_{1/2}$ ITS onto itself is $gs$-homeomorphism. This implies that $sgs$-homeomorphism is both $sgs$-homeomorphism and $gs$-homeomorphism.

b. Every $sgs$-homeomorphism from $T_b$ ITS onto itself is a $sgs$-homeomorphism. This implies that $sgs$-homeomorphism is $gs$-homeomorphism, $sg$-homeomorphism, $sgc$-homeomorphism, and $gs$-homeomorphism.

c. Every $sgs$-homeomorphism from $T_{1/2}$ ITS space onto itself is a $s.h.$ (C.H).

**Proof:** In a $T_{1/2}$ ITS; every $gs$-closed set is a semi-closed set.

d. In a $T_b$ ITS; every $gs$-closed set is closed set.

e. Follows from definition of $T_{1/2}$ space.

**CONCLUSION**

In this paper, we introduce two classes of maps called $sgs$-homeomorphisms and $gs$-homeomorphisms and study their properties. From all of the above statements, we have the following diagram, but the implication appear in the diagram is not seveable.
Summary:
Example 3.6 shows that:
1. \( f \) is sg-homeomorphism, but not sgc-homeomorphism;
2. \( f \) is gs-homeomorphism, but not gsc-homeomorphism and not sgs-homeomorphism;

Example 3.4 shows that; \( f \) is sgs-homeomorphism, but not gsc-homeomorphism.
Example 3.8 shows that; \( f \) is gs-homeomorphism, but not sg-homeomorphism.
Example 3.18 shows that:
1. \( f \) is sgs-homeomorphism, but not gsg-homeomorphism;
2. \( f \) is gsc-homeomorphism, but not gsg-homeomorphism.

Example 4.4 shows that \( f \) is sgs-homeomorphism, but not sgc-homeomorphism, sg-homeomorphism and not gsg-homeomorphism.

REFERENCES