On Soft LC-Spaces and Weak Forms of Soft LC-Spaces

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Abstract
The main purpose of this article is to study the soft LC-spaces as soft spaces in which every soft Lindelöf subset of $\tilde{U}$ is soft closed. Also, we study the weak forms of soft LC-spaces and we discussed their relationships with soft LC-spaces as well as among themselves.

Keywords: Soft $F_{\theta}$-closed set, soft compact space, soft Lindelöf space, soft LC-space, soft $L_i$-space, $i = 1,2,3,4$, soft KC-space, and soft P-space.

1. Preliminaries:
In this paper $P$ is the set of parameters, $U$ is an initial universe set, $P(U)$ is the power set of $U$, and $A \subseteq P$.

Definition (1.1) [1]: A soft set over $U$ is a pair $(H,A)$, where $H$ is a function defined by $H : A \to P(U)$ and $A$ is a non-empty subset of $P$.

Definition (1.2) [5]: A soft set $(H,A)$ over $U$ is called a soft point if there is $e \in A$ such that $H(e) = \{u\}$ for some $u \in U$ and $H(e') = \varnothing$, $\forall e' \in A \setminus \{e\}$ and is denoted by $\tilde{u} = (e,\{u\})$.

Introduction
Molodtsov [1] in 1999 introduced and studied soft set theory as a new mathematical tool for dealing with uncertainty while modeling problems in medical sciences, economics, computer science, engineering physics and social sciences. Shabir and Naz [2] in 2011 investigated the notion of soft topological spaces over an initial universe set with a fixed set of parameters. Molodtsov and et. al. [3] in 2006 and Rong [4] in 2012 introduced and studied soft compact spaces and soft Lindelöf spaces respectively. The main purpose of this paper is to introduce and study a new type of soft spaces called soft LC-spaces and we show that a soft topological space $(U,\tilde{\tau},P)$ is a soft LC-space if and only if each soft point in $\tilde{U}$ has a soft closed neighborhood that is a soft LC-space.

Moreover we discussed weak forms of soft LC-spaces such as soft $L_1$-spaces, soft $L_2$-spaces, soft $L_3$-spaces and soft $L_4$-spaces. The characteristics of these soft spaces and the relation among them also have been studied.

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Definition (1.3)[5]: A soft point \( \bar{u} = (e, \{u\}) \) is called belongs to a soft set \((H, A)\) if \( e \in A \) and \( u \in H(e) \), and is denoted by \( \bar{u} \in (H, A) \).

Definition (1.4)[2]: A soft topology on \( U \) is a family \( \tilde{\tau} \) of soft subsets of \( \tilde{U} \) having the following properties:

(i) \( \tilde{U} \in \tilde{\tau} \) and \( \tilde{\phi} \in \tilde{\tau} \).

(ii) If \((H_1, P), (H_2, P) \in \tilde{\tau} \) \( \Rightarrow \) \((H_1, P) \cap (H_2, P) \in \tilde{\tau} \).

(iii) If \((H_j, P) \in \tilde{\tau} \), \( \forall j \in \Omega \) \( \Rightarrow \) \( \bigcup_{j \in \Omega} (H_j, P) \in \tilde{\tau} \).

The triple \((U, \tilde{\tau}, P)\) is called a soft topological space. The members of \( \tilde{\tau} \) are called soft open sets over \( U \). The complement of a soft open set is called soft closed.

Definition (1.5)[6]: Let \((U, \tilde{\tau}, P)\) be a soft topological space and \( (H, P) \subseteq \tilde{U} \). Then the soft closure of \((H, P)\), denoted by \( cl((H, P)) \) is the intersection of all soft closed sets in \( \tilde{U} \) which contains \((H, P)\).

Definition (1.6)[2]: If \((U, \tilde{\tau}, P)\) is a soft topological space and \( \tilde{\phi} \neq (Y, P) \subseteq \tilde{U} \). The family \( \tilde{\tau}_{(Y, P)} = \{(V, P) \cap (Y, P) : (V, P) \in \tilde{\tau}\} \) is called the relative soft topology on \((Y, P)\) and \((Y, P), \tilde{\tau}_{(Y, P)}, P)\) is called a soft subspace of \((U, \tilde{\tau}, P)\).

Definition (1.7)[7]: A soft topological space \((U, \tilde{\tau}, P)\) is called a soft \( \tilde{T}_1 \)-space if for any two distinct soft points \( \bar{x} \) and \( \bar{y} \) of \( \tilde{U} \), there exists a soft open set in \( \tilde{U} \) containing \( \bar{x} \) but not \( \bar{y} \) and a soft open set in \( \tilde{U} \) containing \( \bar{y} \) but not \( \bar{x} \).

Theorem (1.8)[7]: A soft topological space \((U, \tilde{\tau}, P)\) is a soft \( \tilde{T}_1 \)-space if and only if each soft point in \( \tilde{U} \) is soft closed.

Definition (1.9)[7]: A soft topological space \((U, \tilde{\tau}, P)\) is called a soft \( \tilde{T}_2 \)-space if for any two distinct soft points \( \bar{x} \) and \( \bar{y} \) of \( \tilde{U} \), there are two soft open sets \((H, P)\) and \((K, P)\) in \( \tilde{U} \) such that \( \bar{x} \in (H, P) \), \( \bar{y} \in (K, P) \), and \((H, P) \cap (K, P) = \tilde{\phi} \).

Definition (1.10)[7]: A soft topological space \((U, \tilde{\tau}, P)\) is called a soft regular space if for any soft closed set \((F, P)\) in \( \tilde{U} \) and any soft point \( \bar{x} \) in \( \tilde{U} \) such that \( \bar{x} \in (F, P) \) there exists two soft open sets \((H, P)\) and \((K, P)\) in \( \tilde{U} \) such that \( \bar{x} \in (H, P) \), \( (F, P) \subseteq (K, P) \), and \((H, P) \cap (K, P) = \tilde{\phi} \).

Definition (1.11)[3]: A soft topological space \((U, \tilde{\tau}, P)\) is called soft compact if every soft open cover of \( \tilde{U} \) has a finite soft subcover.

Theorem (1.12)[8]: A soft closed subset of a soft compact space is soft compact.

Theorem (1.13)[9]: A soft compact set in a soft \( \tilde{T}_2 \)-space is soft closed.

Definition (1.14)[4]: A soft topological space \((U, \tilde{\tau}, P)\) is called soft Lindelöf if every soft open cover of \( \tilde{U} \) has a countable soft subcover.

Theorem (1.15)[8]: A soft closed subset of a soft Lindelöf space is soft Lindelöf.
2. Soft LC-Spaces and Weak Forms of Soft LC-Spaces

Now, we introduce and study new types of soft spaces called soft LC-spaces also, we study weak forms of soft LC-spaces such as soft $L_1$-spaces, soft $L_2$-spaces, soft $L_3$-spaces and soft $L_4$-spaces. Further we discussed the equivalent definitions of these soft spaces and the relation among them.

Definition (2.1): A soft topological space $(U, \bar{\tau}, P)$ is called a soft LC-space if every soft Lindelöf subset of $\tilde{U}$ is soft closed.

Definition (2.2): A soft subset $(F, P)$ of a soft topological space $(U, \bar{\tau}, P)$ is called soft $F_\sigma$-closed if it is the soft union of a countable soft closed sets.

Definition (2.3): A soft topological space $(U, \bar{\tau}, P)$ is called a soft P-space if every soft $F_\sigma$-closed set in $\tilde{U}$ is soft closed.

Definition (2.4): A soft topological space $(U, \bar{\tau}, P)$ is called:

(i) A soft $L_1$-space if every soft Lindelöf $F_\sigma$-closed set in $\tilde{U}$ is a soft closed set.

(ii) A soft $L_2$-space if $\text{cl}((L, P))$ is soft Lindelöf whenever $(L, P)$ is a soft Lindelöf set in $\tilde{U}$.

(iii) A soft $L_3$-space if every soft Lindelöf set in $\tilde{U}$ is a soft $F_\sigma$-closed set.

(iv) A soft $L_4$-space if whenever $(L, P)$ is a soft Lindelöf set in $\tilde{U}$, there is a soft Lindelöf $F_\sigma$-closed set $(F, P)$ in $\tilde{U}$ such that $(L, P) \subseteq (F, P) \subseteq \text{cl}((L, P))$.

Theorem (2.5):

(i) If $(U, \bar{\tau}, P)$ is a soft LC-space, then $(U, \bar{\tau}, P)$ is a soft $L_1$-space, $i = 1, 2, 3, 4$.

(ii) If $(U, \bar{\tau}, P)$ is a soft $L_1$-space and a soft $L_3$-space, then $(U, \bar{\tau}, P)$ is a soft LC-space.

(iii) If $(U, \bar{\tau}, P)$ is a soft $L_1$-space and a soft $L_4$-space, then $(U, \bar{\tau}, P)$ is a soft $L_2$-space.

(iv) Every soft $L_2$-space is a soft $L_4$-space and every soft $L_3$-space is a soft $L_4$-space.

(v) Every soft $L_3$-space is a soft $T_1$-space.

(vi) Every soft Lindelöf space is a soft $L_2$-space and every soft $L_2$-space having a soft dense Lindelöf set is soft Lindelöf.

(vii) The property soft $L_3$ is soft hereditary and the properties $L_1$, $L_2$ and $L_4$ are soft hereditary on a soft $F_\sigma$-closed set.

(viii) The property soft LC-space is soft hereditary.

(ix) Every soft P-space is a soft $L_1$-space.

Proof: (i) It is obvious.

(ii) Let $(L, P)$ be a soft Lindelöf set in $\tilde{U}$, since $(U, \bar{\tau}, P)$ is a soft $L_3$-space, then $(L, P)$ is soft $F_\sigma$-closed, but $(U, \bar{\tau}, P)$ is a soft $L_1$-space, then $(L, P)$ is a soft closed set in $\tilde{U}$. Thus $(U, \bar{\tau}, P)$ is a soft LC-space.

(iii) Let $(L, P)$ be a soft Lindelöf set in $\tilde{U}$, since $(U, \bar{\tau}, P)$ is a soft $L_4$-space, then there is a soft Lindelöf $F_\sigma$-closed set $(F, P)$ in $\tilde{U}$ such that $(L, P) \subseteq (F, P) \subseteq \text{cl}((L, P))$. Since $(U, \bar{\tau}, P)$ is a soft $L_1$-space, then $(F, P)$ is soft closed. Hence $\text{cl}((L, P)) \subseteq (F, P) \subseteq \text{cl}((L, P))$, thus $\text{cl}((L, P)) = (F, P)$ is a soft Lindelöf set in $\tilde{U}$. Therefore $(U, \bar{\tau}, P)$ is a soft $L_2$-space.

(iv) Let $(L, P)$ be a soft Lindelöf set in $\tilde{U}$, since $(U, \bar{\tau}, P)$ is a soft $L_2$-space, then $\text{cl}((L, P))$ is soft Lindelöf. Hence $(L, P) \subseteq \text{cl}((L, P)) \subseteq \text{cl}((L, P))$. Since $\text{cl}((L, P))$ is soft closed, then there is $(F, P) = \text{cl}((L, P))$ is a soft $L_1$-space. Thus $(U, \bar{\tau}, P)$ is a soft $L_4$-space. Similarly, we can prove $(U, \bar{\tau}, P)$ is a soft $L_4$-space, if $(U, \bar{\tau}, P)$ is a soft $L_3$-space.

(v) Since $\{\tilde{x}\}$ is a soft Lindelöf set and $(U, \bar{\tau}, P)$ is a soft $L_3$-space, then $\{\tilde{x}\}$ is a soft $F_\sigma$-closed set. Therefore $\{\tilde{x}\}$ is soft closed. Thus $(U, \bar{\tau}, P)$ is a soft $T_1$-space.
(vi) Let \((L, P)\) be a soft Lindelöf set in \(U\), since \(cl((L, P))\) is soft closed in \((U, \tau, P)\) which is a soft Lindelöf space, then by theorem (1.15), \(cl((L, P))\) is soft Lindelöf in \(U\). Thus \((U, \tau, P)\) is a soft L₂-space. Also, if \((U, \tau, P)\) is a soft L₂-space having a soft dense Lindelöf set \(L (P)\), then \(cl((L, P)) = U\). Since \((U, \tau, P)\) is a soft L₂-space, then \((U, \tau, P)\) is soft Lindelöf.

(vii) Let \((U, \tau, P)\) be a soft L₁-space and \((Y, \tau, Y)\) be a soft \(F_\alpha\)-closed subspace of \((U, \tau, P)\). To prove that \((Y, \tau, Y)\) is a soft L₁-space. Let \((A, P)\) be a soft Lindelöf \(F_\alpha\)-closed set in \(Y\). Since \(\tilde{Y} \subset \tilde{U}\), then \((A, P)\) is soft Lindelöf in \(\tilde{U}\) and \(A, P = \bigcup_{n \in N} (F_n, P)\), where \((F_n, P)\) is soft closed in \(\tilde{Y}\).

\[\forall n \in N, \text{ thus } (A, P) = \bigcup_{n \in N} (\tilde{Y} \cap (F_n, P)) = \tilde{Y} \cap \left( \bigcup_{n \in N} (F_n, P) \right), \text{ Where } (F_n, P) \text{ is soft closed in } \tilde{U},\]

\[\forall n \in N. \text{ Since } \tilde{Y} \text{ is a soft } F_\alpha \text{-closed set in } \tilde{U}, \text{ then } \tilde{Y} = \bigcup_{m \in N} (G_m, P), \text{ where } (G_m, P) \text{ is soft closed in } \tilde{U}, \]

\[(G_m, P) \cap (F_n, P) \text{ is soft closed in } \tilde{U}, \text{ thus } (A, P) \text{ is a soft union of a countable soft closed sets in } \tilde{U}, \text{ hence } (A, P) \text{ is a soft } F_\alpha \text{-closed set in } \tilde{U}. \text{ Since } (U, \tau, P) \text{ is a soft L₁-space, then } (A, P) \text{ is soft closed in } \tilde{U} \Rightarrow (A, P) = \tilde{Y} \cap (A, P) = \tilde{Y} \cap \left( \bigcup_{n \in N} (G_m, P) \cap (F_n, P) \right) = \bigcup_{n,m \in N} \tilde{Y} \cap \left( (G_m, P) \cap (F_n, P) \right)\]

is soft closed in \(\tilde{Y}\). Therefore \((Y, \tau, Y, P)\) is a soft L₁-space. Similarly, we can prove other cases.

(viii) It is obvious.

(ix) It is obvious.

**Theorem (2.6):** A soft topological space \((U, \tau, P)\) is a soft LC-space if and only if each soft point in \(\tilde{U}\) has a soft closed neighborhood that is a soft LC-subspace.

**Proof:** If \((U, \tau, P)\) is a soft LC-space, then for each \(\tilde{x} \in \tilde{U}\), \(\tilde{U}\) itself is a soft closed neighborhood that is a soft LC-space. Conversely, let \((L, P)\) be a soft Lindelöf set in \(\tilde{U}\) and let \(\tilde{x} \in (L, P)\). Choose a soft closed neighborhood \((W, P)\tilde{x}\) of \(\tilde{x}\) such that \(((W, P)\tilde{x}, \tau_{(W, p)\tilde{x}}, P)\) is a soft LC-subspace. Then \((W, P)\tilde{x} \cap (L, P)\) is soft Lindelöf in the soft subspace \((W, P)\tilde{x}, \tau_{(W, p)\tilde{x}}, P)\). Since \(((W, P)\tilde{x}, \tau_{(W, p)\tilde{x}})(L, P)\) is a soft LC-space, therefore \((W, P)\tilde{x} \cap (L, P)\) is soft closed in \(((W, P)\tilde{x}, \tau_{(W, p)\tilde{x}})(L, P)\) and so also soft closed in \((U, \tau, P)\). Hence \((W, P)\tilde{x} = (W, P)\tilde{x} \cap (L, P) = (W, P)\tilde{x} - (L, P)\) is a soft open neighborhood of \(\tilde{x}\) in \((W, P)\tilde{x}\) soft disjoint from \(L\), that is \((L, P)\) is soft closed in \((W, P)\tilde{x}\). Thus \((L, P)\) is soft closed in \((U, \tau, P)\).

**Definition (2.7):** A soft topological space \((U, \tau, P)\) is called a soft Q-set space if each soft subset of \(\tilde{U}\) is a soft \(F_\alpha\)-closed set.

**Definition (2.8):** A soft topological space \((U, \tau, P)\) is called a soft hereditarily Lindelöf if each soft subspace of \(\tilde{U}\) is soft Lindelöf.

**Proposition (2.9):** (i) Every soft Q-set space is a soft \(L_3\)-space.

(ii) Every soft hereditarily Lindelöf \(L_3\)-space is a soft Q-set space.

**Proof:** (i) Let \((L, E)\) be a soft Lindelöf set in \(\tilde{U}\), since \((U, \tau, P)\) is a soft Q-set space, then \((L, P)\) is a soft \(F_\alpha\)-closed set. Thus \((U, \tau, P)\) is a soft \(L_3\)-space.

(ii) Let \((L, P)\) be a soft subset of \(\tilde{U}\), since \((U, \tau, P)\) is a soft hereditarily Lindelöf, then \((L, P)\) is a soft Lindelöf set in \((U, \tau, P)\) which is a soft \(L_3\)-space, then \((L, P)\) is a soft \(F_\alpha\)-closed set in \(\tilde{U}\). Hence
(U, \tilde{\tau}, P) is a soft Q-set space.

**Corollary (2.10):**

(i) Every soft Q-set space is a soft \( \tilde{T}_1 \)-space.

(ii) Every soft hereditarily Lindelöf LC-space is a soft Q-set space.

(iii) Every soft \( L_1 \) Q-set space is a soft LC-space.

(iv) Every soft \( L_3 \) P-space is a soft LC-space.

(v) Every soft \( P \) Q-set space is a soft LC-space.

**Proof:** (i) If \((U, \tilde{\tau}, P)\) is a soft Q-set space, then by proposition ((2.9),(ii)), \((U, \tilde{\tau}, P)\) is a soft \( L_3 \)-space, hence \((U, \tilde{\tau}, P)\) is a soft \( \tilde{T}_1 \)-space by theorem ((2.5), (v)).

(ii) Since \((U, \tilde{\tau}, P)\) is a soft LC-space, then by theorem (2.5),(i)), \((U, \tilde{\tau}, P)\) is a soft \( L_3 \)-space, but \((U, \tilde{\tau}, P)\) is a soft hereditarily Lindelöf, then by proposition ((2.9),(ii)), \((U, \tilde{\tau}, P)\) is a soft Q-set space.

(iii) Since \((U, \tilde{\tau}, P)\) is a soft Q-set space, then by proposition ((2.9),(i)), \((U, \tilde{\tau}, P)\) is a soft \( L_3 \)-space, since \((U, \tilde{\tau}, P)\) is a soft \( L_1 \)-space, then \((U, \tilde{\tau}, P)\) is a soft LC-space by theorem (2.5),(ii)).

(iv) It is obvious.

(v) If \((L, P)\) is a soft Lindelöf set in \((U, \tilde{\tau}, P)\) which is a soft Q-set space, then \((L, P)\) is a soft \( F_\sigma \)-closed set, but \((U, \tilde{\tau}, P)\) is a soft P-space, so \((L, P)\) is a soft closed set. Hence \((U, \tilde{\tau}, P)\) is a soft LC-space.

**Corollary (2.11):** Every soft \( L_1 \) Q-set space is soft hereditary, \( i = 1, 2, 4 \).

**Proof:** This is obvious by theorem (2.5), (vii)) and definition (2.7).

**Corollary (2.12):** For a soft hereditarily Lindelöf P-space \((U, \tilde{\tau}, P)\) the following statements are equivalent:

(i) \((U, \tilde{\tau}, P)\) is a soft LC-space.

(ii) \((U, \tilde{\tau}, P)\) is a soft Q-set space.

**Proof:** (i) \(\Rightarrow\) (ii): This is obvious by proposition ((2.9),(ii)).

(ii) \(\Rightarrow\) (i): This is obvious by corollary (2.10,v).

**Definition (2.13):** A soft topological space \((U, \tilde{\tau}, P)\) is called a soft KC-space if every soft compact subset of \( \tilde{U} \) is soft closed.

**Proposition (2.14):**

(i) Every soft \( \tilde{T}_2 \)-space is a soft KC-space.

(ii) Every soft KC-space is a soft \( \tilde{T}_1 \)-space.

**Proof:** It is obvious.

**Proposition (2.15):**

(i) Every soft LC-space is a soft KC-space.

(ii) Every soft LC-space is a soft \( \tilde{T}_1 \)-space.

**Proof:** It is obvious.

**Theorem (2.16):** Every soft \( \tilde{T}_2 \) P-space \((U, \tilde{\tau}, P)\) is a soft LC-space.

**Proof:** Let \((L, P)\) be a soft Lindelöf set in \( \tilde{U} \). To prove that \((L, P)\) is soft closed in \( \tilde{U} \). Let \( \tilde{x} \in (L, P)^c \Rightarrow \forall y, \tilde{y} \in (L, P) \), we get \( \tilde{x} \neq \tilde{y} \), since \((U, \tilde{\tau}, P)\) is a soft \( \tilde{T}_2 \)-space, then \( \exists (H, P) \tilde{x} \) and \((K, \tilde{P}) \tilde{y} \) are soft open sets in \( \tilde{U} \) such that \( \tilde{x} \in (H, P) \tilde{x}, \tilde{y} \in (K, \tilde{P}) \tilde{y} \) and \((H, P) \tilde{x} \bigcap (K, \tilde{P}) \tilde{y} = \emptyset \).

Hence \((L, P) \subseteq \bigcup_{\tilde{y} \in (L, P)} \tilde{y} \), thus \( \{(K, \tilde{P}) \tilde{y}, \tilde{y} \in (L, P)\} \) is a soft open cover of \((L, P)\). Since \((L, P)\) is soft Lindelöf \( \Rightarrow \exists \{(K, \tilde{P}) \tilde{y} \}_{\tilde{y} \in (L, P)} \) is a countable soft subcover of \((L, P)\). Let \((W, P) = \bigcup_{\tilde{y} \in (L, P)} \tilde{y} \) and \((V, P) = \bigcap_{\tilde{y} \in (L, P)} \tilde{y} \Rightarrow (W, P) \) is soft open, since it is a soft union of soft open sets.
open sets and \((V,P)\) is also soft open, since \((U,\tilde{\tau},P)\) is a soft P-space and soft intersection of a countable soft open sets is soft open. Hence \(\tilde{x} \in (V,P)\) and \((L,P) \subseteq (W,P)\). To prove that

\[\text{Corollary (2.17): Every soft } \tilde{T}_1 \text{-regular P-space is a soft LC-space.}\]

\[\text{Proof:}\] Let \(\tilde{x}, \tilde{y} \in \tilde{U}\) such that \(\tilde{x} \neq \tilde{y}\). Since \((U,\tilde{\tau},P)\) is a soft \(\tilde{T}_1\)-space, then by theorem (1.8), \(\{\tilde{x}\}\) is soft closed in \(\tilde{U}\) and \(\tilde{y} \notin \tilde{x}\). Since \((U,\tilde{\tau},P)\) is a soft regular space, then by definition (1.10), \(\exists (H,P)\) and \((K,P)\) are soft open sets in \(\tilde{U}\) such that \(\{\tilde{x}\} \subseteq (H,P)\), \(\tilde{y} \notin (K,P)\) and \((H,P) \cap (K,P) = \emptyset\). Hence \(\exists (H,P)\) and \((K,P)\) are soft open sets in \(\tilde{U}\) such that \(\tilde{x} \in (H,P)\), \(\tilde{y} \notin (K,P)\) and \((H,P) \cap (K,P) = \emptyset\) \(\Rightarrow (U,\tilde{\tau},P)\) is a soft \(\tilde{T}_2\)-space, since \((U,\tilde{\tau},P)\) is a soft P-space, then by theorem (2.16), \((U,\tilde{\tau},P)\) is a soft LC-space.

**Proposition (2.18):** Countable soft union of soft Lindelöf sets is soft Lindelöf.

**Proof:** Let \(\{\{A_n,P\}\}_{n \in N}\) be a countable family of soft Lindelöf sets in \(\tilde{U}\). To prove that

\[\bigcup_{n \in N}(A_n,P)\]

is soft Lindelöf. Let \(\{(V_{nm},P)\}_{m \in M}\) be any soft open cover of \(\bigcup_{n \in N}(A_n,P)\). To prove that \(\{(V_{nm},P)\}_{m \in M}\) is soft open cover of \((A_n,P)\), \(\forall n \in N\). Since \((A_n,P)\) is soft Lindelöf \(\forall n \in N \exists \{V_{nm},P\}_{m \in M}\) is a countable soft subcover \(\forall n \in N\). That is \((A_n,P) \subseteq \bigcup_{m \in N}(V_{nm},P), \forall n \in N \Rightarrow \bigcup_{n \in N}(A_n,P) = \bigcup_{m \in M}(V_{nm},P)\). Since union of countable family of countable sets is countable

\[\Rightarrow \{\{(V_{nm},P)\}_{n \in N}, \{m \in M\}\} \text{ is a countable soft subcover of } \bigcup_{n \in N}(A_n,P) \Rightarrow \bigcup_{n \in N}(A_n,P)\] is soft Lindelöf.

**Proposition (2.19):** For a soft Lindelöf \(\tilde{T}_2\)-space \((U,\tilde{\tau},P)\) the following statements are equivalent:

(i) \((U,\tilde{\tau},P)\) is a soft LC-space.

(ii) \((U,\tilde{\tau},P)\) is a soft P-space.

**Proof:** (i) \(\Rightarrow\) (ii): Let \((A,P)\) be a soft \(F_\sigma\)-closed set in \(\tilde{U}\) \(\Rightarrow (A,P) = \bigcup_{n \in N}(F_n,P)\), where \((F_n,P)\) is soft closed in \(\tilde{U}\), \(\forall n \in N\). Since \((U,\tilde{\tau},P)\) is soft Lindelöf, then by theorem (1.15), \((F_n,P)\) is soft Lindelöf in \(\tilde{U}\), \(\forall n \in N\), hence by proposition (2.18), \((A,P) = \bigcup_{n \in N}(F_n,P)\) is soft Lindelöf in \(\tilde{U}\), but \((U,\tilde{\tau},P)\) is a soft LC-space, then \((A,P)\) is soft closed in \(\tilde{U}\). Thus \((U,\tilde{\tau},P)\) is a soft P-space.

(ii) \(\Rightarrow\) (i): This is obvious by theorem (2.16).

**Proposition (2.20):** For a soft Lindelöf Q-set space \((U,\tilde{\tau},P)\) the following statements are equivalent:

(i) \((U,\tilde{\tau},P)\) is a soft LC-space.

(ii) \((U,\tilde{\tau},P)\) is a soft P-space.

**Proof:** (i) \(\Rightarrow\) (ii): This is obvious by proposition (2.19).

(ii) \(\Rightarrow\) (i): This is obvious by corollary (2.10,v).

**Proposition (2.21):** For a soft regular P-space \((U,\tilde{\tau},P)\) the following statements are equivalent:

(i) \((U,\tilde{\tau},P)\) is a soft LC-space.
(ii) \((U, \tau, P)\) is a soft KC-space.

(iii) \((U, \tau, P)\) is a soft \(\tilde{T}_1\)-space.

**Proof:** (i) \(\rightarrow\) (ii): This is obvious by proposition (2.15),(i).

(ii) \(\rightarrow\) (i): Let \((U, \tau, P)\) be a soft KC-space, then by proposition ((2.14), (ii)), \((U, \tau, P)\) is a soft \(\tilde{T}_1\)-space, but \((U, \tau, P)\) is soft regular, then \((U, \tau, P)\) is a soft \(\tilde{T}_2\)-space. Since \((U, \tau, P)\) is a soft P-space, then by theorem (2.16), \((U, \tau, P)\) is a soft LC-space.

(ii) \(\rightarrow\) (iii): This is obvious by proposition (2.14),(ii).

(iii) \(\rightarrow\) (ii): Let \((U, \tau, P)\) be a soft \(\tilde{T}_1\)-space, since \((U, \tau, P)\) is soft regular, then \((U, \tau, P)\) is a soft \(\tilde{T}_2\)-space. Hence \((U, \tau, P)\) is a soft KC-space by proposition ((2.14), (i)).

**Definition (2.22):** A soft topological space \((U, \tau, P)\) is called a soft \(R_1\)-space if \(\bar{x}\) and \(\bar{y}\) have disjoint soft neighborhoods whenever \(\text{cl}([x]) \not\subset \text{cl}([y])\). Clearly a soft space is soft \(\tilde{T}_2\) if and only if its soft \(\tilde{T}_1\) and soft \(R_1\).

**Theorem (2.23):** For a soft \(R_1\)-space \((U, \tau, P)\) the following statements are equivalent:

(i) \((U, \tau, P)\) is a soft KC-space.

(ii) \((U, \tau, P)\) is a soft \(\tilde{T}_1\)-space.

(iii) \((U, \tau, P)\) is a soft \(\tilde{T}_2\)-space.

**Proof:** (i) \(\rightarrow\) (ii): This is obvious by proposition ((2.14),(iii)).

(ii) \(\rightarrow\) (i): Let \((U, \tau, P)\) be a soft \(\tilde{T}_1\)-space, since \((U, \tau, P)\) is a soft \(R_1\)-space, then \((U, \tau, P)\) is a soft \(\tilde{T}_2\)-space by definition (2.22). So \((U, \tau, P)\) is a soft KC-space by proposition ((2.14),(i)).

(ii) \(\rightarrow\) (iii): This is obvious by definition (2.22).

(iii) \(\rightarrow\) (ii): It is obvious.

**Corollary (2.24):** (i) Every soft \(R_1\)KC-space is a soft \(\tilde{T}_2\)-space.

(ii) Every soft \(R_1\)Q-set space is a soft \(\tilde{T}_2\)-space.

(iii) Every soft \(R_1\)Q-set space is a soft KC-space.

(iv) Every soft \(L_1\) \(L_3\)-space is a soft KC-space.

(v) Every soft \(R_1\) \(L_3\)-space is a soft \(\tilde{T}_2\)-space.

(vi) Every soft \(R_1\) \(L_3\)-space is a soft KC-space.

**Proof:** It is obvious.

**Corollary (2.25):** For a soft regular space \((U, \tau, P)\) the following statements are equivalent:

(i) \((U, \tau, P)\) is a soft KC-space.

(ii) \((U, \tau, P)\) is a soft \(\tilde{T}_1\)-space.

**Proof:** (i) \(\rightarrow\) (ii): This is obvious by proposition ((2.14),(ii)).

(ii) \(\rightarrow\) (i): Let \((U, \tau, P)\) be a soft \(\tilde{T}_1\)-space, since \((U, \tau, P)\) is soft regular, then \((U, \tau, P)\) is a soft \(\tilde{T}_2\)-space. Hence by proposition ((2.14),(i)), \((U, \tau, P)\) is a soft KC-space.

**Theorem (2.26):** For a soft \(\tilde{T}_2\)-space \((U, \tau, P)\) the following statements are equivalent:

(i) \((U, \tau, P)\) is a soft LC-space.

(ii) \((U, \tau, P)\) is a soft \(L_1\)-space and a soft \(L_2\)-space.

**Proof:** (i) \(\rightarrow\) (ii): This is obvious by theorem ((2.5),(i)).

(ii) \(\rightarrow\) (i): Let \((L, P)\) be a soft Lindelöf set in \(U\) and \(\bar{x} \in (L, P)\). To prove that \(\bar{x} \in \text{cl}((L, P))\). Since \((U, \tau, P)\) is a soft \(\tilde{T}_2\)-space, then \(\forall \tilde{y} \in (L, P), \exists (V, P)\tilde{y} \in \tilde{r} \) such that \(\tilde{y} \in (V, P)\tilde{y}\) and
\( x \in \text{cl}(V, P) \). Hence \( \{ (V, P) : y \in (L, P) \} \) is a soft open cover of \((L, P)\). Since \((L, P)\) is soft Lindelöf \( \Rightarrow \exists \{ (V, P) y \in \bigcap_{n \in N} \text{cl}(V, P) y_n \} \) is a countable soft subcover of \((L, P)\). Thus \((L, P) \subseteq \bigcup_{n \in N} (V, P) y_n \subseteq \bigcup_{n \in N} \text{cl}(V, P) y_n \). For each \( n \in N \), \((L, P) \bigcap \text{cl}(V, P) y_n \) is soft Lindelöf.

Since \((U, \tau, P)\) is a soft \( L_2 \)-space, \((U, \tau, P) \bigcap \text{cl}(V, P) y_n \) is soft Lindelöf. If \((W, P) = \bigcup_{n \in N} \text{cl}(V, P) y_n \), then \((W, P)\) is soft Lindelöf \( F_\sigma \)-closed set in \( U \), but \((U, \tau, P)\) is a soft \( L_1 \)-space, then \((W, P)\) is soft closed and \( x \in \text{cl}(W, P) \), hence \( x \in \text{cl}(L, P) \). Thus \((L, P)\) is a soft closed set in \( U \). Therefore \((U, \tau, P)\) is a soft LC-space.

**Corollary (2.27):** For a soft Lindelöf \( T_2 \)-space \((U, \tau, P)\) the following statements are equivalent:

(i) \((U, \tau, P)\) is a soft P-space.

(ii) \((U, \tau, P)\) is a soft LC-space.

(iii) \((U, \tau, P)\) is a soft \( L_1 \)-space and a soft \( L_2 \)-space.

**Proof:** This is obvious by proposition (2.19) and theorem (2.26).

**Corollary (2.28):** For a soft regular \( L_1 \)-space \((U, \tau, P)\) the following statements are equivalent:

(i) \((U, \tau, P)\) is a soft LC-space.

(ii) \((U, \tau, P)\) is a soft \( T_1 \)-space.

**Proof:** (i) \( \Rightarrow \) (ii): This is obvious by proposition ((2.15),(iii)).

(ii) \( \Rightarrow \) (i): This is obvious by theorem (2.26).

**Corollary (2.29):** For a soft \( R_1 \)-space \((U, \tau, P)\) the following statements are equivalent:

(i) \((U, \tau, P)\) is a soft LC-space.

(ii) \((U, \tau, P)\) is a soft \( T_1 \)-space.

**Proof:** (i) \( \Rightarrow \) (ii): This is obvious by proposition ((2.15), (iii)).

(ii) \( \Rightarrow \) (i): This is obvious by theorem (2.26).

**Corollary (2.30):** For a soft discrete space \((U, \tau, P)\) the following statements are equivalent:

(i) \((U, \tau, P)\) is a soft LC-space.

(ii) \((U, \tau, P)\) is a soft \( L_2 \)-space.

**Proof:** It is obvious.

**Theorem (2.31):** If \((U, \tau, P)\) is a soft topological space and \( Y \subseteq U \), \( Y = \bigcup_{i=1}^{n} Y_i \), where \( Y_i \), \( i = 1, 2, \ldots, n \) are soft closed LC-subspaces of \( U \), then \( Y \) is a soft LC-subspace.

**Proof:** Let \((L, P)\) be a soft Lindelöf subset of \( Y \), then \( \bigcap_{i=1}^{n} Y_i \subseteq (L, P) \). n are soft closed in \((L, P)\) which is soft Lindelöf so \( \bigcap_{i=1}^{n} Y_i \subseteq (L, P) \). i = 1, 2, \ldots, n are soft LC-subspaces, then \( \bigcap_{i=1}^{n} Y_i \subseteq (L, P) \) is a soft closed in \( \bigcap_{i=1}^{n} Y_i \), i = 1, 2, \ldots, n. Since \( \bigcap_{i=1}^{n} Y_i \), i = 1, 2, \ldots, n is soft LC-subspace, then \( \bigcap_{i=1}^{n} Y_i \subseteq (L, P) \) is a soft closed in \( \bigcap_{i=1}^{n} Y_i \), i = 1, 2, \ldots, n. Since \( \bigcap_{i=1}^{n} Y_i \), i = 1, 2, \ldots, n is soft closed in \( U \), then \( \bigcap_{i=1}^{n} Y_i \subseteq (L, P) \), i = 1, 2, \ldots, n is soft closed in \( U \). But \((L, P) = \bigcup_{n=1}^{n} (Y_i \subseteq (L, P)) \) so \((L, P)\) is soft closed in \( U \) and also in \( Y \). Hence \( Y \) is a soft LC-subspace.

**Proposition (2.32):** Every soft Lindelöf \( L_1 \)-space is a soft P-space.

**Proof:** Let \((A, P)\) be a soft \( F_\sigma \)-closed set in \( U \) \( \Rightarrow \) \((A, P) = \bigcup_{n \in N} (F_n, P)\), where \((F_n, P)\) is soft closed in \( U \). Since \((U, \tau, P)\) is soft Lindelöf, then by theorem (1.15), \((F_n, P)\) is soft Lindelöf in
\[ \tilde{U}, \forall n \in \mathbb{N}. \text{Hence } (A, P) = \bigcup_{n \in \mathbb{N}} (F_n, P) \text{ is soft Lindelöf in } \tilde{U} \text{ by proposition (2.18). Since } (U, \tilde{\tau}, P) \text{ is a soft } L_1-\text{space, then } (A, P) \text{ is soft closed in } \tilde{U}. \text{ Thus } (U, \tilde{\tau}, P) \text{ is a soft P-space. }
\]

**Proposition (2.33):** For a soft Lindelöf space \( (U, \tilde{\tau}, P) \) the following statements are equivalent:

(i) \( (U, \tilde{\tau}, P) \) is a soft LC-space.

(ii) \( (U, \tilde{\tau}, P) \) is a soft \( L_1 \)-space.

**Proof:** (i) \( \rightarrow \) (ii): This is obvious by theorem ((2.5), (i)).

(ii) \( \rightarrow \) (i): Let \( (U, \tilde{\tau}, P) \) be a soft \( L_1 \)-space, since \( (U, \tilde{\tau}, P) \) is a soft Lindelöf space, then by proposition (2.32), \( (U, \tilde{\tau}, P) \) is a soft P-space. Since \( (U, \tilde{\tau}, P) \) is a soft \( \tilde{T}_2 \)-space, then \( (U, \tilde{\tau}, P) \) is a soft LC-space by theorem (2.16).

**Proposition (2.34):** For a soft Q-set space \( (U, \tilde{\tau}, P) \) having a soft dense Lindelöf subset the following statements are equivalent:

(i) \( (U, \tilde{\tau}, P) \) is a soft P-space.

(ii) \( (U, \tilde{\tau}, P) \) is a soft Lindelöf and a soft \( L_1 \)-space.

**Proof:** (i) \( \rightarrow \) (ii): If \( (U, \tilde{\tau}, P) \) is a soft P-space, then \( (U, \tilde{\tau}, P) \) is a soft \( L_1 \)-space. Since \( (U, \tilde{\tau}, P) \) is a soft Q-set space, then by corollary ((2.10),(iii)), \( (U, \tilde{\tau}, P) \) is a soft LC-space, hence \( (U, \tilde{\tau}, P) \) is a soft \( L_2 \)-space. Since \( (U, \tilde{\tau}, P) \) having a soft dense Lindelöf subset, then by theorem ((2.5),(vi)), \( (U, \tilde{\tau}, P) \) is a soft Lindelöf space.

(ii) \( \rightarrow \) (i): This is obvious by proposition (2.32).

**Proposition (2.35):** For a soft \( \tilde{T}_2 \)-space \( (U, \tilde{\tau}, P) \) having a soft dense Lindelöf subset the following statements are equivalent:

(i) \( (U, \tilde{\tau}, P) \) is a soft LC-space.

(ii) \( (U, \tilde{\tau}, P) \) is a soft Lindelöf space.

**Proof:** (i) \( \rightarrow \) (ii): If \( (U, \tilde{\tau}, P) \) is a soft LC-space, then \( (U, \tilde{\tau}, P) \) is a soft \( L_2 \)-space. Since \( (U, \tilde{\tau}, P) \) having a soft dense Lindelöf subset, then by theorem (2.5), (vi), \( (U, \tilde{\tau}, P) \) is a soft Lindelöf space.

(ii) \( \rightarrow \) (i): Let \( (U, \tilde{\tau}, P) \) be a soft Lindelöf space, then by theorem ((2.5),(vi)), \( (U, \tilde{\tau}, P) \) is a soft \( L_2 \)-space. Since \( (U, \tilde{\tau}, P) \) is a soft \( \tilde{T}_2 \)-space, then by theorem (2.26), \( (U, \tilde{\tau}, P) \) is a soft LC-space.

**Proposition (2.36):** For a soft Lindelöf Q-set space \( (U, \tilde{\tau}, P) \) the following statements are equivalent:

(i) \( (U, \tilde{\tau}, P) \) is a soft \( L_1 \)-space.

(ii) \( (U, \tilde{\tau}, P) \) is a soft \( L_2 \)-space and a soft P-space.

**Proof:** (i) \( \rightarrow \) (ii): Let \( (U, \tilde{\tau}, P) \) be a soft \( L_1 \)-space, since \( (U, \tilde{\tau}, P) \) is a soft Lindelöf space, then by proposition (2.32), \( (U, \tilde{\tau}, P) \) is a soft P-space. Since \( (U, \tilde{\tau}, P) \) is a soft Q-set space, then by corollary ((2.10),v), \( (U, \tilde{\tau}, P) \) is a soft \( L_2 \)-space.

(ii) \( \rightarrow \) (i): This is obvious by theorem ((2.5),(ix)).

**Theorem (2.37):** If \( (U, \tilde{\tau}, P) \) is a soft topological space and \( \tilde{Y} \subseteq \tilde{U} \), \( \tilde{Y} = \bigcup_{i=1}^{n} \tilde{Y}_i \), where \( \tilde{Y}_i \), \( i = 1, 2, \ldots, n \) are soft closed \( L_2 \)-subspaces of \( \tilde{U} \), then \( \tilde{Y} \) is a soft \( L_2 \)-subspace.

**Proof:** Let \( (L, P) \) be a soft Lindelöf subset of \( \tilde{Y} \), then \( \tilde{Y}_i \cap (L, P), i = 1, 2, \ldots, n \) are soft closed in \( (L, P) \) which is soft Lindelöf, so \( \tilde{Y}_i \cap (L, P), i = 1, 2, \ldots, n \) are soft Lindelöf subset of \( \tilde{Y}_i \), \( i = 1, 2, \ldots, n \). Since \( \tilde{Y}_i \), \( i = 1, 2, \ldots, n \) is a soft \( L_2 \)-subspace, then \( \operatorname{cl}(\tilde{Y}_i \cap (L, P)) \) is a soft Lindelöf in \( \tilde{Y}_i \), \( i = 1, 2, \ldots, n \). Hence \( \operatorname{cl}((\tilde{Y}_i \cap (L, P)), i = 1, 2, \ldots, n \) is soft Lindelöf in \( \tilde{Y} \). But
\[ \text{cl}((L, P)) = \bigcup_{i=1}^{n} \text{cl}(\bigcap_{i=1}^{n} (Y_i \cap (L, P))) \]

so \( \text{cl}((L, P)) \) is soft Lindelöf in \( \bar{Y} \). Hence \( \bar{Y} \) is a soft \( L_2 \)-subspace.

**Theorem (2.38):** For a soft \( R_1 \)-P-space \( (U, \bar{\tau}, P) \) the following statements are equivalent:

(i) \((U, \bar{\tau}, P)\) is a soft LC-space.

(ii) \((U, \bar{\tau}, P)\) is a soft KC-space.

(iii) \((U, \bar{\tau}, P)\) is a soft \( \bar{T}_1 \)-space.

(iv) \((U, \bar{\tau}, P)\) is a soft \( \bar{T}_2 \)-space.

(v) \((U, \bar{\tau}, P)\) is a soft \( L_3 \)-space.

**Proof:** (i) \(\Rightarrow\) (ii): This is obvious by proposition ((2.15),(i)).

(ii) \(\Rightarrow\) (i): Let \((U, \bar{\tau}, P)\) be a soft KC-space, since \((U, \bar{\tau}, P)\) is a soft \( R_1 \)-space, then by proposition ((2.14),(iii)) and definition (2.22), \((U, \bar{\tau}, P)\) is a soft \( \bar{T}_2 \)-space, since \((U, \bar{\tau}, P)\) is a soft P-space, then by theorem (2.16), \((U, \bar{\tau}, P)\) is a soft LC-space.

(ii) \(\Rightarrow\) (iii): This is obvious by proposition ((2.14),(iii)).

(iii) \(\Rightarrow\) (ii): Let \((U, \bar{\tau}, P)\) be a soft \( \bar{T}_1 \)-space, since \((U, \bar{\tau}, P)\) is a soft \( R_1 \)-space, then \((U, \bar{\tau}, P)\) is a soft \( \bar{T}_2 \)-space by definition (2.22). So \((U, \bar{\tau}, P)\) is a soft KC-space by proposition ((2.14),(i)).

(iii) \(\Rightarrow\) (iv): Let \((U, \bar{\tau}, P)\) be a soft \( \bar{T}_1 \)-space, since \((U, \bar{\tau}, P)\) is a soft \( R_1 \)-space, then \((U, \bar{\tau}, P)\) is a soft \( \bar{T}_2 \)-space by definition (2.22).

(iv) \(\Rightarrow\) (iii): It is obvious.

(iv) \(\Rightarrow\) (v): Let \((U, \bar{\tau}, P)\) be a soft \( \bar{T}_2 \)-space, since \((U, \bar{\tau}, P)\) is a soft P-space, then \((U, \bar{\tau}, P)\) is a soft LC-space by theorem (2.16), so \((U, \bar{\tau}, P)\) is a soft \( L_3 \)-space.

(v) \(\Rightarrow\) (iv): Let \((U, \bar{\tau}, P)\) be a soft \( L_3 \)-space, then \((U, \bar{\tau}, P)\) is a soft \( \bar{T}_1 \)-space. Since \((U, \bar{\tau}, P)\) is a soft \( R_1 \)-space, then by definition (2.22), \((U, \bar{\tau}, P)\) is a soft \( \bar{T}_2 \)-space.

**Theorem (2.39):** For a soft \( \bar{T}_2 \) \( L_1 \)-space \((U, \bar{\tau}, P)\) the following statements are equivalent:

(i) \((U, \bar{\tau}, P)\) is a soft LC-space.

(ii) \((U, \bar{\tau}, P)\) is a soft \( L_4 \)-space.

(iii) \((U, \bar{\tau}, P)\) is a soft \( L_3 \)-space.

(iv) \((U, \bar{\tau}, P)\) is a soft \( L_2 \)-space.

**Proof:** (i) \(\Rightarrow\) (ii): This is obvious by theorem ((2.5),(i)).

(ii) \(\Rightarrow\) (i): This is obvious by theorem ((2.5),(iii)) and theorem (2.26).

(ii) \(\Rightarrow\) (iii): Let \((U, \bar{\tau}, P)\) be a soft \( L_4 \)-space, since \((U, \bar{\tau}, P)\) is a soft \( L_1 \)-space, then \((U, \bar{\tau}, P)\) is a soft \( L_2 \)-space by theorem ((2.5),(iii)). Since \((U, \bar{\tau}, P)\) is a soft \( \bar{T}_2 \)-space, then by theorem (2.26), \((U, \bar{\tau}, P)\) is a soft LC-space. Hence \((U, \bar{\tau}, P)\) is a soft \( L_3 \)-space by theorem ((2.5),(i)).

(iii) \(\Rightarrow\) (ii): This is obvious by theorem ((2.5),(iv)).

(iii) \(\Rightarrow\) (iv): Let \((U, \bar{\tau}, P)\) be a soft \( L_3 \)-space, since \((U, \bar{\tau}, P)\) is a soft \( L_1 \)-space, then by theorem ((2.5),(iii)), \((U, \bar{\tau}, P)\) is a soft LC-space. Hence \((U, \bar{\tau}, P)\) is a soft \( L_2 \)-space by theorem ((2.5),(i)).

(iv) \(\Rightarrow\) (iii): Let \((U, \bar{\tau}, P)\) be a soft \( L_2 \)-space, since \((U, \bar{\tau}, P)\) is a soft \( \bar{T}_2 \) \( L_1 \)-space, then \((U, \bar{\tau}, P)\) is a soft LC-space by theorem (2.26). Hence \((U, \bar{\tau}, P)\) is a soft \( L_3 \)-space by theorem ((2.5),(i)).
References