Another type of separation axioms

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Abstract:

The main purpose of this paper is to study properties of semi-closed(open) sets s-closed (s-open)sets and generalized-closed(open) sets g-closed(g-open) sets and study the relationship between them. And also introduce another type of separation axioms called it S-separation axiom and the characteristics that can be preserved of some type of functions on many spaces as S-T₀, S-T₁, S-T₂ and we explain that the property of S-T₀, S-T₁, S-T₂ are topological property if the function is injective and s*-open.

1- Introduction:

The concepts of semi closed(open) sets s-closed(s-open) sets are introduction by Levine.N in 1963 [7]. He defined a set A in a topological space X to be s-open set if for some set G, \( G \subseteq A \subseteq cl(A) \), where \( cl(A) \) denoted to the closure of a in X. A set F is s-closed if it’s complement is s-open set.

In 1970 Levine .N [ 8] introduced another concepts called it generalized closed (open) sets g-closed(g-open) sets in order to extend many of the important properties of closed set to larger family.

In 1971 Crossieg .S.G and Hildebrand .S.K , introduced the the concept of semi closure and they define it, the semi closure of a set A in a topological space X is the smallest semi closed (s-closed) set containing A [4], and denoted it by \( scl(A) \).
In 1982 Malgahan introduce g-closed function's concept and give some theorems of preservation of normality and regularity [9].

In this paper we continue to study of s-closed set and study the relation between s-closed set and g-closed set, and proved they are independent concepts, and give a new notation of semi separation axioms (S-separation axiom) and characteristic that can be preserved of some type of function on many spaces, as S-T₀, S-T₁, S-T₂. Also we will study the relation between the usual separation axiom and S-separation axiom.

2- Preliminaries:

In this section we give definitions, remarks, examples and also we prove some results on s-closed(s-open) sets.

Definition (2-1): [7]

Let X be a topological space, A ⊆ X, A is called semi closed set (s-closed set) if there exist F closed set in X such that

\[ \text{Int}(F) \subseteq A \subseteq F \]

Where \( \text{Int}(F) \) denoted to interior of F.

Definition (2-2): [4]

Let X be a topological space, A ⊆ X, A is called semi closed set (s-closed set) if

\[ \text{Int}(\text{cl}(A)) \subseteq A \]

Remark (2-3): [5]

The two above definitions (2-1),(2-2) are equivalence.

Remark (2-4): [1]

The complement of semi closed(s-closed) set is called semi open(s-open) set.

Remark (2-5): [1]

Every open set is s-open set but the converse may be not true as following example.

Example (2-5):

Let \( X = \{1,2,3\} \) , \( T = \{\emptyset, X, \{1\}, \{1,2\} \} \) is topology on X.

Clear the s-open set in X are \( \{\emptyset, X, \{1\}, \{1,2\}, \{1,3\} \} \).

Hence \{1,3\} is s-open set but it is not open set.

Definition (2-6): [8]

Let (X,T) be a topological space, A ⊆ X is called g-closed set if

\[ \text{cl}(A) \subseteq O \]

whenever \( A \subseteq O \), O is open set in X. The complement of g-closed set is g-open set.
Remark (2-7) :
Every closed set is g-closed set but the convers my not be true as the following example .

Example (2-8):
Let $X=\{a,b,c,d\}$ , $T = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c,d\}\} , $ is topology on $X$. Let $A=\{c\}$ , $cl(A) = \{c,d\}$ , hence $A$ is g-closed set but it is not closed set .

Remark (2-9) :
The intersection of two s-open set not may be s-open set as the following example:

Example (2-10):
Let $(R, T_u)$ is usual topology . Clear that $(0,5]$ and $[2,7)$ are two s-open sets in $(R, T_u)$ , but it’s intersection $[2,5]$ is not s-open set in $(R, T_u)$.

Theorem (2-11): [6]
If $A$ is s-open set in $X$ and $U$ is open in $X$ then $U \cap A$ is s-open in $U$.

Theorem (2-12):
IF $F$ is closed set and $B$ is s-closed set in $X$ ,then $F \cup B$ is s-closed set.

Proof: Let $(X,T)$ be a topological space ,and let $F$ is closed set and $B$ is s-closed set in $X$ . $\therefore F^c$ is open set in $X$ , $B^c$ is s-open set in $X$ . then $F^c \cap B^c$ is s-open set in $X$ [ theorem 2-11]

But $F^c \cap B^c = (F \cup B)^c$ Demorgan’s law. $\therefore (F \cup B)^c$ is s-open in $X$ .

Hence $F \cup B$ is s-closed set in $X$ (Remark (2-4)) .

Definition(2-13):[1]
A function $f : X \rightarrow Y$ is called:

a- S-open (S-closed ) function if $\forall G \subseteq X$ is open (closed ) then $f(G) \subseteq Y$ is s-open (s-closed).

b- $S^*$-open ($S^*$-closed) function if $\forall U \subseteq X$ is s-open (s-closed )

then $f(U) \subseteq Y$ is open (closed).

c- $S^{**}$-open ($S^{**}$-closed) function if $\forall U \subseteq X$ is s-open (s-closed )

then $f(U) \subseteq Y$ is s-open (s-closed).

3- The Main Result:

In this section we will study the relation between the concepts s-closed and g-closed sets and we will show that there are two
independent concepts completely through the following four examples:

The following example show that there exist g-closed set and s-closed set at the same time.

**Example (3-1)**:

Show in example (2-8) the set \( A = \{c\} \), \( cl(A) = \{c,d\} \), hence \( A \) is g-closed set, and it’s s-closed set because if \( F = \{c,d\} \) is closed set, then \( int(F) = \{\phi\} \Rightarrow \phi \subseteq \{c\} \subseteq \{c,d\} \), \( \therefore \) \( int(F) \subseteq A \subseteq F \)

Hence \( A = \{c\} \) is g-closed set and s-closed set at the same time.

The following examlpe show that there exist some sets are g-closed set but it is not s-closed set.

**Example (3-2)**:

Let \( X = \{a,b,c\} \), \( T = \{\phi, X, \{a\}, \{a,b\}\} \) be topology on \( X \).
Let \( A = \{a,c\} \), \( A \) is g-closed set but it is not s-closed set.

The following examlpe explain that there exist some sets are s-closed set but it is not g-closed set.

**Example (3-3)**:

Let \( (\mathbb{R}, T_u) \) be the usual topology on \( \mathbb{R} \), let \( A = [0,1) \) clearly \( A \) is s-closed set but it is not g-closed set because

If \( O = (-1,1) \) open interval in \( \mathbb{R} \), show that \( A \subseteq O \) but \( cl(A) \not\subset O \)

Hence \( A \) is not g-closed set.

The following examlpe show that there exist some sets that are not s-closed and are not g-closed set.

**Example (3-4)**:

In the example (2-8), let \( A = \{a,b\} \), clear that \( cl(A) = X \), \( int(cl(A)) = X \) show that \( A \) is not s-closed set because \( int(cl(A)) = X \subset A \).

And it is not g-closed set too because \( A \subseteq \{a,b\} \) but \( cl(A) = X \subset A \).

Hence we can say that the s-closed set and g-closed set are independent completely conscepts.

**4- S –\( T_0 \) space**

**Definition (4-1):**

A topolgical space \( X \) is called S-\( T_0 \) space if and only if for each \( x \) and \( y \) are distinct points in \( X \), there exist an s-open set \( W \) in \( X \) containing one of them and not the other.

**Remark (4-2):**

Every \( T_0 \) space is S-\( T_0 \), but the converse may not be true as the following example.
Example (4-3):
Let $X=\{1,2,3\}$ and let $T=\{\emptyset, X, \{1\}\}$ be a topology on $X$, clear $(X,T)$ is $S-T_0$ space because the s-open sets in $X$ are: $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$, hence every two distinct point in $X$, there exist s-open set in $X$ containing one of them and not the other, and it is not $T_0$ space. Note that $b$ and $c$ are two different points in $X$, and we can’t find an open set in $X$ which contains one of them and not the other. Then $X$ is $S-T_0$ space but it is not $T_0$ space.

**Theorem (4-4):**
Every open subspace of $S-T_0$ space is $S-T_0$ space.

**Proof**
Let $Y$ be an open subspace of $S-T_0$ space $X$, and let $x, y$ be two distinct point of $Y$, then there exist an s-open set $A$ in $X$ containing one of them and not the other, let it be containing $x$ but not $y$ then $A\cap Y$ is s-open set in $Y$ [theorem(2-11)] containing $x$ but not $y$.

Hence $Y$ is $S-T_0$ space.

**Theorem (4-5):**
Let $f : X \to Y$ be injective function and $S^*$- open function,
If $X$ is $S-T_0$ space then $Y$ is $S-T_0$ space.

**Proof**
Let $y_1, y_2$ be two distinct points in $Y$, since $f$ is injective function then there exist two distinct points $x_1, x_2$ in $X$ such that $y_1 = f(x_1), y_2 = f(x_2)$, but $X$ is $S-T_0$ space and $x_1, x_2$ are two distinct point in it, then there exist g-open set $V$ in $X$ containing one of them and not the other (i.e $x_1 \in V, x_2 \notin V$) then $f(x_1) \in f(V)$ and $f(x_2) \notin f(V)$.

Since $f$ is $S^*$- open function and $V$ is s-open set in $X$, then $f(V)$ is open set in $Y$. \therefore $f(V)$ is s-open set (remark 2-5), since $y_1 = f(x_1), y_2 = f(x_2)$ then $y_1 \in f(V), y_2 \notin f(V)$. Hence $Y$ is $S-T_0$ space.

**Corollary (4-6):**
$S-T_0$ is a topological property where the function is injective and $S^*$-open function.
5- S-T₁ space

Definition (5-1):
A topological space X is called S-T₁ space if and only if for each
x and y are distinct point in X there exist two s-open sets
G₁, G₂
in X such that x ∈ G₁, y \notin G₁ and x \notin G₂, y ∈ G₂.

Remarks (5-2):
(a) Every S-T₁ space is S-T₀ space.
(b) Every T₁ space is S-T₁ space, but the converse
may not be true as in the following example.

Example (5-3):
Let X= {1,2,3}, T = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} be topology on X,
then the space X is S-T₁ but not T₁ space.

Theorem (5-4):
A topological space X is S-T₁ space if and only if
singleton {x} is s-closed set in X.

Proof:
Let X be S-T₁ space and let x ∈ X, to prove that {x} is s-closed set
we will prove X-{x} is s-open set in X, let y ∈ X-{x} → x \neq y ∈ X,
and since X is S-T₁ space then their exist two s-open sets G₁, G₂
such that x \notin G₁, y ∈ G₂ ⊆ X-{x}. since y ∈ G₂ ⊆ X-{x}
then X-{x} is s-open set, Hence {x} is s-closed set.

⇐ conversely:
Let x \neq y ∈ X then {x},{y} are s-closed sets,
\text{i.e.} X-{x} is s-open set clearly x \notin X-{x} and y ∈ X-{x}.
similarly X-{y} is s-open set, y \notin X-{y} and x ∈ X-{y}.
Hence X is S-T₁ space.

Theorem (5-5):
Let X be S-T₁ space and f : X → Y be an injective function and
S*- open function then Y is S-T₁ space.

Proof:
Let y₁, y₂ be two distinct points in Y, since f is injective function,
then there exist two distinct points x₁, x₂ in X, such that y₁=f(x₁),
y₂=f(x₂), but X is S-T₁ space and x₁, x₂ are distinct points in it, then
there exist two s-open sets V₁, V₂ in X such that x₁ ∈ V₁, x₂ \notin V₁
and x₂ ∈ V₂, x₁ \notin V₂.
\text{i.e} f(x₁) ∈ f(V₁), f(x₂) \notin f(V₁) \text{ and } f(x₂) ∈ f(V₂), f(x₁) \notin f(V₂).
Since f is S*-open function and V₁, V₂ are two s-open set in X
then f(V₁), f(V₂) are two open sets in Y (Def 2-13-b).
Hence f(V₁) and f(V₂) are s-open sets in Y, but y₁=f(x₁) and
\[ y_2 = f(x_2) \text{ then } y_1 \in f(V_1), \ y_2 \not\in f(V_2) \text{ and } y_2 \in f(V_2), \ y_1 \not\in f(V_2). \]

Hence \( Y \) is S-T₁ space.

**Corollary (5-6):**

S-T₁ is a topological property where the function is injective and \( S^* \)-open function.

**Theorem (4-7):**

Every open subspace of a S-T₁ space is S-T₁ space.

**Proof:**

Let \( X \) be S-T₁ space and let \( G \) be an open subspace of \( X \),

let \( x \in G \), since \( X \) is S-T₁ space, \( X-\{x\} \) is s-open set in \( X \).

\( G \cap (X-\{x\}) = G-\{x\} \) and it is s-open set in \( G \) [Theorem 2-11]

Then \( \{x\} \) is s-closed in \( G \).

Hence \( G \) is S-T₁ space [Theorem 5-4].

6- **S – T₂ space ((S – Hausdorff space))**

**Definition (6-1):**

A topological space \( X \) is called S-T₂ space (S-Hausdorff),

if \( \forall x_1 \neq x_2 \in X, \ \exists \text{ two s-open sets } H_1, H_2 \text{ in } X \)
such that \( x_1 \in H_1 \) and \( x_2 \in H_2 \) and \( H_1 \cap H_2 = \emptyset \).

**Remarks (6-2):**

1- Every S-T₂ space is S-T₁ space.

2- Every T₂ space is S-T₂ space, but the converse may not be true as the following example.

**Example (6-3):**

Let \( X=\{a,b,c\}, \ T_i \) is indiscrete topology on \( X \), then ( \( X, T_i \)) is S-T₂ space but it is not T₂-space.

**Theorem (6-4):**

Every open subspace of S-T₂ space is S-T₂ space.

**Proof:**

By the same method in proving theorem (5-7)

**Remark (6-5):**

Every singleton subset of S-T₂ space is S-closed set.

**Theorem (6-6):**

Let \( X \) be S-T₂ space and \( f : X \to Y \), be injective, \( S^* \)-open
function, where $X$ and $Y$ are topological spaces then $Y$ is S-T$_2$ space.

**Proof:**
Let $y_1, y_2$ be two distinct points in $Y$, since $f$ is injective function, then there exist two distinct points $x_1, x_2$ in $X$, such that $y_1 = f(x_1)$, $y_2 = f(x_2)$, but $X$ is S-T$_2$ space and $x_1, x_2$ are distinct points in it, then there exist two s-open sets $H_1, H_2$ in $X$ such that $x_1 \in H_1$, $x_2 \in H_2$ and $H_1 \cap H_2 = \emptyset$. Since $f$ is S*-open function then $f(H_1)$ and $f(H_2)$ are S-open sets in $Y$, but $y_1 = f(x_1)$ then $y_1 \in f(H_1)$, since $x_2 \in H_2$ then $f(x_2) \in f(H_2)$, but $y_2 = f(x_2)$ then $y_2 \in f(H_2)$, to prove that $f(H_1) \cap f(H_2) = \emptyset$, since $H_1 \cap H_2 = \emptyset$ then $f(H_1) \cap f(H_2) = \emptyset$. Hence $Y$ is S-T$_1$ space.

**Corollary (6-7):**
S-T$_2$ is topological property where the function is injective and S*-open function.

**Proof:**
Clear from theorem (6-6)

The following diagram shows the relation between usual separation axiom and S-separation axiom:

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T_0  ──────── T_1  ──────── T_2
  ^              ^              ^
  |              |              |
S-T_0  ──────── S-T_1  ──────── S-T_2
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References:

3- Bohn .E and Leejong ,semi topological groups Amer.Math. monthly 72 , 1965
5- DAS.P, Note on some application on semi open sets progress mathematics,Al ahabod University 1975.
11-Noiri . Generalized $\theta$-closed set of almost para compact space, jour , of Math and comp ScI(Mathser) vol 9 no 2, 1996.
نوع آخر من بديعات الفصل

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المستخلص

إن الهدف الرئيس لهذا العمل هو دراسة خواص أنماط أخرى من المجموعات المغلقة (المفتوحة) وهي المجموعات المغلقة-س (المفتوحة-س) أي شبه المغلقة (شبه المفتوحة) والمجموعات المغلقة-ج (المفتوحة-ج) وتقديم عدد من المبرهنات والنتائج والملاحظات والأمثلة الخاصة بذلك ودراسة العلاقة بين هذين النمطين وقد بينا بأنهما مستقلان تماماً.

ثم أيضاً تقدم نوع آخر جديد من بديعات الفصل يسمى بديعات الفصل-س ودراسة خواص هذا المفهوم وتقديم بعض الفضائات الخاصة به ومنها ودراسة العلاقة فيه بينهما وعلاقتها مع بديعات الفصل الاعتيادية، وتم أيضاً دراسة تأثير بعض الدوال الخاصة بالمحافظة على هذه الخاصية ومنها الدالة المفتوحة-س، وقد بينا أيضاً بان خاصية تبولوجية إذا كانت الدالة متباينة S-T0، S-T1، S-T2 وضعيفة-س ومفتوحة-س

كلمات مفتاحية:

مجموعات شبه مغلقة (مفتوحة)، مجموعات مغلقة-س (مفتوحة-س)، بديعات فصل-س،