DERIVATION OF STIFFNESS MATRIX FOR A GENERAL TWO DIMENSIONAL CURVED ELEMENT IN GENERAL GLOBAL COORDINATES SYSTEM

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ABSTRACT

In the present paper, the derivation of stiffness matrix for a general two dimensional curved element in global coordinates system is presented. The derivation depends on the assumption that any curved in-plane element can be approximated by a specified curve in polar coordinates. The polar curve assumed in this paper depends on some variables that enable it to represent any two dimensional curved element. The derivation process accompanied by complex integrals which are evaluated by using (Gaussian Quadrature) method of numerical integration. One numerical example is presented to verify the accuracy and efficiency of the derived stiffness matrix. The verification contains a comparison with the results of the exact solution. Very good agreement is obtained between the results of the derived stiffness matrix and the results of the exact model.

الخلاصة

في هذا البحث تم ابتكار مصفوفة الاجسام لأي عنصر موجز ناقل الأبعاد، ويتم نظام الإحداثيات العام، للاستخدام. يعتمد على رضية أن أي عنصر موجز ناقل الأبعاد يمكن أن يقرب منحنى معين بالاختلافات النقطية، المتجهي النقلي المفرود في هذا البحث، يستند على بعض المقارنات التي تمكنه من تحليل أي عنصر موجز ناقل الأبعاد، عملية الانتقال صرح بالتكاملات معقدة والتي تم حسابها باستخدام طريقة (Gaussian Quadrature) المتجهي ناقل الأبعاد، مقارنة النتائج مع نتائج الاحتكاكات القوية الجديدة. تم مقارنة النتائج في تحسينها مع الحل الموجز، ومن خلال المقارنة تم الحصول على نتائج قريبة جداً من الحل الموجز وتفوق لأزيد عن (7,1%).

KEYWORDS
Stiffness matrix; two dimensional element; general curved element; global coordinates
INTRODUCTION
In the past, the curve beam or arch represents one of the few structural systems which make it possible to cover large spans. The earliest inhabitants developed the arch as an important element of their architecture as expressed by remaining bridges, aqueducts and large public buildings. To-day, the same importance is presented especially in bridge construction. Typical forms of curved beams are: circular arches, cantilever-curved beams, elliptical arches, cubic, square arches and catenary curved beams. Most of early modern curved beams have semicircular shapes. Now, curved beams or arched structures are constructed in different shapes and from variable materials as brick, steel, reinforced concrete, ferrocement and timber. The main aim of the curved beam is to enhance the load-carrying capacity, which may come from stiffening behavior due to membrane action. A literature survey indicates that a substantial amount of works that deal with the analysis of arches by using circular curved finite element. Just(1982) presented the exact (6x6) stiffness matrix for a circularly curved beam subjected to loading in its own plane. This matrix was derived from the governing differential equations and from the finite element procedure. The strain energy contributed axial and flexural actions were considered. Akhtar(1987) expressed the stiffness matrix of a single circular member of uniform cross-section. He also obtained the fixed end actions due to concentrated load acting at any point on the member making any angle with radial direction at this point. The effect of shear deformations was neglected. Litewka and Rakowski(1998) derived the exact stiffness matrix for a curved beam element with constant curvature(circular curved elements). The plane two node six-degree-of-freedom element was considered. Hadji(2002) developed a circular curved beam element stiffness matrix. He included the effect of shear deformations. The derived matrix is used in the nonlinear analysis of reinforced concrete circular arches. It is clear from the preceding review that there is no formulation of a more general stiffness matrix for a curved element including circular and non-circular curved elements. The objective of the present paper is to derive a more general stiffness matrix that deals with circular and non-circular curved elements by using the principle of the strain energy. In addition, the derived stiffness matrix is in global coordinates system and can be applied on any curved element without any transformation.

- Derivation of Stiffness Matrix For a General Curved Beam Element:
Before the derivation of stiffness matrix, a general equation for any curved in-plane element must be found. In this paper, a general equation in polar coordinates is suggested to represent any plane curved element. The suggested equation is:

\[ r = a \cos n\theta \]  

(1)

Which represents a family of (flower-shaped) curves or roses depending on the value of \( n \) equally spaced petals of radius \( a \) at \( \theta = 0 \). So, if one takes the first quadrant of eq.(1), any plane curved element can be fitted by choosing a specified values for \( n \) and \( a \) depends on span length and a number of known \((x,y)\) coordinates for the curve. For \( n=1 \), the curve represents a circle as shown in Fig.1 and for \( n=2 \) represents a rose of \((2n)\) equally spaced petals if \( n \) is even \((n=2,4,6,...)\) and of \( n \) equally spaced petals if \( n \) is odd \((n=3,5,7,...)\). As \( n \) reaches high values, the above eq.(1) represent a straight beam. So, the chosen polar curve represents an infinite number of curves varies from a circle \((n=1)\) and the degree of curvature will decrease with the increment of \( n \) until it will reach to a straight element \((n=\infty)\) or large values. The above explanation can be seen in Fig.1.
Fig.1: The Graph of $r = a \cos n\theta$

The derivation of the stiffness matrix depends on the principle of the strain energy. The model forces and displacements of a curved element are shown in Fig.2. The internal forces can be expressed in terms of the nodal forces at node (1) by using the static equilibrium equations. The global coordinate system is considered for the directions of the nodal forces and displacements. Hence, by using the suggested polar equation, the following internal forces can be obtained

$$P = P \cos \beta + Q_1 \sin \beta$$

(2)

Where ($\beta$) is the inclination angle of the tangent to the polar curve at a point having coordinates ($r, \theta$) which can be expressed as

$$\tan \beta = \frac{dy}{dx} = \frac{dy}{d \theta} \frac{d \theta}{dx} = \frac{n \sin n \theta \sin \theta - \cos n \theta \cos \theta}{r \sin n \theta \cos \theta + \cos n \theta \sin \theta}$$

(3)

In which $x = r \cos \theta, y = r \sin \theta, r = a \cos n \theta$

$$M = Q_1 (r \cos \theta - r^2 \cos \theta_2) - (P_1 (r \sin \theta - r^2 \sin \theta_2) + M_1)$$

(4)
Fig. 2: The curved Element Considered In This Study
(a) Orientation of The Element (b) Force and Displacement Systems on The Element

The strain energy used here can be expressed as

\[ U = \frac{1}{2EI} \int \theta_2 \dot{\theta}_2 M^2 ds + \frac{1}{2AF} \int p^2 ds \]  \hspace{1cm} (5)

Where \( ds = \rho \sqrt{\left( \cos \theta \right)^2 + \left( \sin \theta \right)^2} \, d\theta \) is the length of the curve segment in polar coordinates used to find the strain energy along the curve. Anton et al (2002). By substituting expressions (2) and (4) into the equation of the strain energy (5) one can get

\[ U = \frac{1}{2EI} \int \frac{1}{A} \left( \frac{Q_j^2(a_j)}{1} - 2Q_j P_j(a_j) - 2Q_j M_j(a_j) + \rho_1^2(a_1) + 2j M_j(a_j) + M_j^2(a_j) \right) \]  \hspace{1cm} (6)

Where

\[ a_1 = \frac{\partial}{\partial \theta_1} \left[ \frac{\cos \theta}{2} \left( \frac{\left( \cos \theta \right)^2 + \left( \sin \theta \right)^2 \right) \right] d\theta \]  \hspace{1cm} (7)

\[ a_2 = \frac{\partial}{\partial \theta_2} \left[ \frac{\cos \theta}{2} \left( \frac{\left( \cos \theta \right)^2 + \left( \sin \theta \right)^2 \right) \right] d\theta \]  \hspace{1cm} (8)

\[ a_3 = \frac{\partial}{\partial \theta_3} \left[ \frac{\left( \cos \theta \right)^2}{2} + \frac{\left( \sin \theta \right)^2}{2} \right] d\theta \]  \hspace{1cm} (9)

\[ a_4 = \frac{\partial}{\partial \theta_4} \left[ \frac{\left( \cos \theta \right)^2}{2} - \frac{\left( \sin \theta \right)^2}{2} \right] d\theta \]  \hspace{1cm} (10)
\[ e_5 = \int_0^\theta \left[ \frac{a^2 (\cos \theta \sin \theta) - a^2 \sin \theta}{\left( a^2 + a^2 \sin \theta \right)^2} \right] d\theta \] (11)

\[ e_6 = \int_0^\theta \left[ \frac{a^2 \cos \theta}{\left( a^2 + a^2 \sin \theta \right)^2} \right] d\theta \] (12)

\[ e_7 = \int_0^\theta \left[ \frac{a^2 \cos \theta}{\left( a^2 + a^2 \sin \theta \right)^2} \right] d\theta \] (13)

\[ e_8 = \int_0^\theta \left[ \frac{a^2 \sin \theta}{\left( a^2 + a^2 \sin \theta \right)^2} \right] d\theta \] (14)

\[ e_n = \int_0^\theta \left[ \frac{a^2 \sin \theta}{\left( a^2 + a^2 \sin \theta \right)^2} \right] d\theta \] (15)

Integrations of eqs. (7 to 15) is complicated, so, it can be found by using **Gaussian-Quadrature** method of numerical integration (see Appendix - A). It is found that three Gaussian points give close results to the exact solution for the example presented in this paper. The stiffness coefficients corresponding to the degrees of freedom shown in Fig.2(b) can be obtained by using **Castigliano's second theorem** BORSEI and SCHMIDT (2003), which states that the deflection caused by an external force is equal to the partial derivative of the strain energy (U) with respect to that force.

The partial derivatives of the strain energy (U) with respect to \( P_1 \), \( Q_1 \) and \( M_1 \) respectively are:

\[ \frac{\partial U}{\partial P_1} = \frac{1}{2EI} \left[ P_1 (c_1) + Q_1 (c_2) + M_1 (c_3) \right] \] (16)

\[ \frac{\partial U}{\partial Q_1} = \frac{1}{2EI} \left[ P_1 (c_2) + Q_1 (c_4) + M_1 (c_5) \right] \] (17)

\[ \frac{\partial U}{\partial M_1} = \frac{1}{2EI} \left[ P_1 (c_3) + Q_1 (c_5) + M_1 (c_6) \right] \] (18)

Where

\[ c_1 = 2(a_4) + \frac{2aI}{A} (a_7) \]

\[ c_2 = \frac{aI}{A} (a_8) - 2(a_2) \]

\[ c_3 = 2(a_5) \]

\[ c_4 = 2(c_1) + \frac{2aI}{A} (a_9) \]

\[ c_5 = -2(a_3) \]

\[ c_6 = 2(a_6) \] (19)

The stiffness coefficient \( k_{ij} \) can be defined as the force of type \( i \) which required to cause a unit displacement of type \( j \) with all other types of displacements equal to zero. Therefore, equations (16), (17) and (18) will be used to find the stiffness matrix of the element.

### 2.1 Axial Stiffness

Consider the element shown in Fig.3 which is subjected to a unit axial displacement. The stiffness coefficients corresponding to that displacement can be obtained by setting expressions (16), (17) and (18) equal to 1, 0 and 0 respectively, hence

\[ P_1 (c_1) + Q_1 (c_2) + M_1 (c_3) = 2EI \] (20)
Derivation of Stiffness Matrix for
A general Two Dimensional Curved
Element in General Global
Coordinates System

\[ P_1(e_2) + Q_1(e_4) + M_1(e_5) = 0 \]  
(21)
\[ P_1(e_3) + Q_1(e_5) + M_1(e_6) = 0 \]  
(22)

Fig. 3: Curved Element Subjected to a Unit Axial Displacement

Solving eqs. (20), (21) and (22) simultaneously yields
\[ P_1 = \frac{2EI}{A_1} \]  
(23)
\[ Q_1 = \frac{2EI}{A_1} Z_1 \]  
(24)
\[ M_1 = \frac{-2EI}{A_1} \left[ \frac{c_3 + c_5 Z_1}{c_6} \right] \]  
(25)

Where
\[ A_1 = \left[ c_1 + c_3 Z_1 - \frac{c_3}{c_6} \left( c_3 + c_5 Z_1 \right) \right], Z_1 = \frac{c_5 c_3 - c_2}{c_6 - c_5^2} \]

From equilibrium requirements
\[ P_j = -P_1 = \frac{-2EI}{A_1} \]  
(26)
\[ Q_j = -Q_1 = \frac{-2EI}{A_1} Z_1 \]  
(27)
\[ M_j = Q_j L - (P_1 L_1 + M_1) \]
\[ = \frac{2EI}{A_1} \left[ Z_1 L - \left( L_1 - \frac{c_3 + c_5 Z_1}{c_6} \right) \right] \]  
(28)

* Lateral Stiffness
Proceeding as in the previous section, stiffness coefficients due to a unit lateral displacement at a
node \((i)\) (Fig. 4) can be found by making expressions (15), (17) and (18) equal to 0.1 and 0
respectively, hence
\[ P_j(c_1) + Q_j(c_2) + M_j(c_3) = 0 \]  
\[ P_j(c_2) + Q_j(c_3) + M_j(c_5) = 2EI \]  
\[ P_j(c_3) + Q_j(c_5) + M_j(c_6) = 0 \]

Again, by solving the above three equations simultaneously, the following expressions can be obtained

\[ \eta_j = -\frac{2EI}{A_2} \left[ \frac{c_2 + c_3 \eta_j}{c_1} \right] \]  
\[ Q_j = \frac{2EI}{A_2}, \]  
\[ M_j = \frac{2EI}{A_2} \eta_j \]  

Fig.4: Curved Element Subjected to a Unit Lateral Displacement

Where

\[ A_2 = \left[ c_4 + c_5 \eta_j - \frac{c_2}{c_1} \left( \frac{c_5 c_2 - c_4}{c_1} \right) \right] \]

\[ \eta_j = \frac{c_5 c_2 - c_4}{c_6 - \frac{c_2}{c_1}} \]

Also, due to equilibrium requirements

\[ P_j = \frac{2EI}{A_2} \left[ \frac{c_2 + c_3 \eta_j}{c_1} \right] \]  
\[ Q_j = \frac{-2EI}{A_2} \]  
\[ M_j = \frac{2EI}{A_2} \left[ L - \left( \frac{c_2 + c_3 \eta_j}{c_1} \right) \right] \]

- Rotational Stiffness

Stiffness coefficients corresponding to a unit rotational displacement at a node \((i)\) (Fig.5) can be found by setting expressions \((16),(17)\) and \((18)\) equal to 0, 0 and 1 respectively, hence

\[ P_j(c_1) + Q_j(c_2) + M_j(c_3) = 0 \]  
\[ P_j(c_2) + Q_j(c_4) + M_j(c_5) = 0 \]  
\[ P_j(c_3) + Q_j(c_5) + M_j(c_6) = 2EI \]

Solving eqs.\((38),(39)\), and \((40)\) simultaneously, one can get
Fig. 5: Curved Element Subjected to a Unit Rotational Displacement

\[ p_i = \frac{-2EI}{A_3} \left[ \frac{c_2Z_3 + c_3}{c_1} \right] \]  
\[ q_i = \frac{2EI}{A_3} Z_3 \]  
\[ m_i = \frac{2EI}{A_3} \]  
In which

\[ A_3 = \left[ c_6 + c_5 Z_3 - \frac{c_3}{c_1} (c_3 + c_2 Z_3) \right] \]
\[ Z_3 = \frac{c_1}{c_4 - c_2 c_5} \]

From equilibrium requirements, the following expressions can be obtained

\[ p_j = \frac{2EI}{A_3} \left[ \frac{c_2Z_3 + c_3}{c_1} \right] \]  
\[ q_j = \frac{-2EI}{A_3} Z_3 \]  
\[ m_j = q_j L - (p_j L_1 + m_1) \]  
\[ = \frac{2EI}{A_3} \left[ Z_3 L - (1 - \frac{c_2 Z_3 + c_3}{c_1}) L_1 \right] \]

All other stiffness coefficients can be found from symmetry and equilibrium requirements. The coefficients of the \((6 \times 6)\) stiffness matrix in the global coordinates system according to the degrees of freedom shown in Fig. 2 (b) is as follows:

\[ k_{11} = \frac{2EI}{A_1} \]  
\[ k_{21} = \frac{2EI}{A_1} Z_1 \]  
\[ k_{31} = \frac{-2EI}{A_1} \left[ \frac{c_3 + c_5 Z_1}{c_6} \right] \]  
\[ k_{41} = \frac{-2EI}{A_1} \]
\[ K_{51} = \frac{-2EI}{A_1} Z_1 \]

\[ K_{61} = \frac{2EI}{A_1} \left[ Z_1 L - \left( L_1 - \left( \frac{c_3 + c_5 Z_1}{c_6} \right) \right) \right] \]

\[ K_{22} = \frac{2EI}{A_1} \]

\[ K_{32} = \frac{2EI}{A_2} Z_2 \]

\[ K_{42} = \frac{2EI}{A_2} \left[ \frac{c_2 + c_3 Z_2}{c_1} \right] \]

\[ K_{52} = \frac{-2EI}{c_2} \]

\[ K_{62} = \frac{2EI}{A_2} \left[ L - (Z_2 - \left( \frac{c_2 + c_3 Z_2}{c_1} \right) L_1) \right] \]

\[ K_{33} = \frac{2EI}{A_1} \]

\[ K_{43} = \frac{2EI}{A_1} \left[ \frac{c_2 Z_3 + c_3}{c_1} \right] \]

\[ K_{53} = \frac{-2EI}{A_3} Z_3 \]

\[ K_{63} = \frac{2EI}{A_3} \left[ Z_3 L - \left( L_1 - \left( \frac{c_2 Z_3 + c_3}{c_1} \right) L_1 \right) \right] \]

\[ K_{44} = \frac{2EI}{A_1} \]

\[ K_{54} = \frac{2EI}{A_1} Z_1 \]

\[ K_{54} = \frac{-2EI}{A_1} \left[ Z_1 L - \left( L_1 - \left( \frac{c_3 + c_5 Z_1}{c_6} \right) \right) \right] \]

\[ K_{55} = \frac{2EI}{A_2} \]

\[ K_{65} = \frac{2EI}{A_2} \left[ L - (Z_2 - \left( \frac{c_2 + c_3 Z_2}{c_1} \right) L_1) \right] \]

\[ K_{66} = \left[ K_{61} L - (K_{61} L_1 + K_{63}) \right] \]

The above coefficients can be written in a matrix form as follows

\[
\begin{bmatrix}
K_{11} & K_{21} & K_{31} & K_{41} & K_{51} & K_{61} \\
K_{21} & K_{22} & K_{32} & K_{42} & K_{52} & K_{62} \\
K_{31} & K_{32} & K_{33} & K_{43} & K_{53} & K_{63} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{54} & K_{64} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{65} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66}
\end{bmatrix}
\]
- Numerical Example
To verify the validity and efficiency of the derived matrix, a cantilever-curved beam shown in Fig.6 is analyzed by using the derived stiffness matrix. The curved beam is analyzed previously by Just (1982) by using an exact circularly curved beam element derived by him. In this element, the strain energy contributed by axial and flexural actions was considered. Three subtended angles (γ) of the curved cantilever were investigated. The beam was analyzed by using a single curved element and also by approximating it into a various numbers of equal straight segments. In the present study, the end displacements of point (B), (UB, VB) are found by using the derived stiffness matrix for the three values of (γ). The results are listed in Table (1). Through the comparison of the two solutions, it is found that the results obtained by using the derived stiffness matrix is close to the exact solution obtained by Just (1982).

![Fig.6: Curved Cantilever of The Numerical Example](image)

In the above figure, the unknowns (r1, r2, φ1, φ2) can easily found from the geometry of the curved beam.

<table>
<thead>
<tr>
<th>Table (1): Comparison of Results of The Numerical Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>End Deflections (mm)</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>vB</td>
</tr>
<tr>
<td>uB</td>
</tr>
<tr>
<td>vB</td>
</tr>
<tr>
<td>uB</td>
</tr>
<tr>
<td>vB</td>
</tr>
<tr>
<td>uB</td>
</tr>
</tbody>
</table>
- CONCLUSIONS
The derived stiffness matrix is found to be efficient for the analysis of curved beams as shown by the comparison of results obtained from the present analysis and the exact solution. It is found that the difference between the present and the exact analyses is not more than (1.7%). The derived stiffness matrix can be used for a wide range of curved elements starting from circular curved elements to straight elements. In addition, the global coordinates system is considered in the derivation of stiffness matrix, so, it can be used without any transformation. This will offer an economical time and fewer calculations solution than the derivations presented by different researchers.

- REFERENCES

- NOTATIONS
The following symbols are used in this paper
A: cross-sectional area of the element
E: modulus of elasticity
l: the moment of inertia about the major axis
Ky: coefficients of stiffness matrix \([K] \)
L: the horizontal projection length of the curved element on \(x\)-axis (span length)
L1: the vertical projection length of the curved element on \(x\)-axis
M: internal moment at a point with \((r, \theta)\) coordinates on the curved element
P: internal axial force at a point with \((r, \theta)\) coordinates on the curved element
Q: internal shear force at a point with \((r, \theta)\) coordinates on the curved element
ui,uj: horizontal displacements at the node \((i)\) and \((j)\) respectively
vi,vj: vertical displacements at the node \((i)\) and \((j)\) respectively
Appendix-A (Numerical Integration)

One of the most formulas which is used for numerical integration is (Gaussian Quadrature) formula. To estimate the value of the integration \( \int_{\theta_1}^{\theta_2} f(\theta) d\theta \) according to (Gaussian Quadrature) formula, the interval of the integration will be changed from \([\theta_1, \theta_2]\) to \([-1,1]\) by a suitable transformation of variable. Let the new variable (\(\alpha\)), where \(-1 \leq \alpha \leq 1\), be defined by

\[
\alpha = \frac{2\theta - (a + b)}{b - a}
\]

(A-1)

Also define a new function \(F(\alpha)\) so that

\[
F(\alpha) = f(\theta) = f\left(\frac{(b - a)\alpha + (b + a)}{2}\right)
\]

(A-2)

Then, integration of \(F(\alpha)\) between the integration limit \([-1,1]\) can be found as follows

\[
\int_{-1}^{1} F(\alpha) d\alpha = \sum_{i=1}^{n} w_i F(\alpha_i)
\]

(A-3)

Where \(n\) is the number of Gaussian points. So, according to eq.(A-3) and by the substitutions

\[
\theta = \frac{(a + b) + (b - a)\alpha}{2}, \quad \text{and}, \quad d\theta = \frac{(b - a)}{2} d\alpha
\]

\[
\int_{a}^{b} f(\theta) d\theta = \frac{(b - a)}{2} \left[ \sum_{i=1}^{n} w_i f\left(\frac{(a + b) + (b - a)\alpha_i}{2}\right) \right]
\]

(A-4)

Where \((w_i)\) is the weight factor and \((\alpha_i)\) is the corresponding base point.

**Example:**

Find \(A = \int_{\pi/4}^{\pi/2} (\cos \theta)^4 d\theta\)

(a) Exact solution

\[
A = \left[ \frac{1}{4} (\cos \theta)^3 \sin \theta + \frac{3}{4} \left( \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right) \right]_{\pi/4}^{\pi/2}
\]

\[= 0.04447\]

(b) Approximate solution

Using three Gaussian points, the corresponding base point and weight factors are:
\[ \alpha_1 = 0.0 \quad w_1 = 0.8888888 \quad \theta_1 = 1.178 \\
\alpha_2 = +0.77459 \quad w_2 = 0.5555555 \quad \theta_2 = 1.482 \\
\alpha_3 = -0.77459 \quad w_3 = 0.5555555 \quad \theta_3 = 0.874 \\
\]

Hence,
\[
A = \frac{1}{\pi} \int_{\pi/4}^{\pi/2} (\cos \theta)^{\alpha} \sin \theta \, d\theta = \frac{(b-a)}{2} \left[ \sum_{i=1}^{3} w_i (\cos \theta_i)^{\alpha} \right] = 0.3927 \left[ 0.888888(0.021456) + 0.555555(0.16969) \right]
\]
\[ A = 0.0445 \]

It can be seen that the two solutions are very close to each other.

Table (A-1): The values of the appropriate base points and the corresponding weight factors for

\[ n = 1,2,3,\ldots,6 \] points formula

<table>
<thead>
<tr>
<th>Roots (( \alpha_i ))</th>
<th>( \int_{-1}^{1} F(\alpha) , d\alpha = \sum_{i=1}^{n} w_i F(\alpha_i) )</th>
<th>Weight factors (( w_i ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>+0.57735, 02691 89626</td>
<td>Two-point Formula</td>
<td>1.00000000000000000000</td>
</tr>
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<td>1.00000000000000000000</td>
<td>0.88888888888888888888</td>
</tr>
<tr>
<td>0.000000000000000000</td>
<td>Three-point Formula</td>
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