Characterizing Internal and External Sets

Dr. Tahir H. Ismail*  Hind Y. Saleh**  Barah M. Sulaiman***

"تمييز المجموعات الداخلية والخارجية"

الملخص

الهدف من هذا البحث هو إعطاء تمييز بين المجموعات الخارجية والمجموعات الداخلية وبعض العلاقات بين الكالكسيات والهالات، ومن أهم النتائج التي حصلنا عليها:

- تكون المجموعة $G$ كالكسي إذا وفقط إذا وجدت متتالية متزايدة بدقة من المجموعات الداخلية $T_n$ بحيث أن $G = \bigcup_{n \in \mathbb{N}} T_n$.

كذلك تكون المجموعة $H$ هالة إذا وفقط إذا وجدت متتالية متناقصة بدقة من المجموعات الداخلية $S_n$ بحيث أن $H = \bigcap_{n \in \mathbb{N}} S_n$.

- إذا كانت هالة و كالكسي $G$ في المجموعة $H$ بحيث $G \subseteq H$.

- إذا كانت $G \neq H$ كالكسي فإن $H$ هالة.

- إذا كانت $G$ كالكسي فإن المجموعة $G''$ لكل الدوال الداخلية هالة.

$\Rightarrow$ إذا وجد $f : G \to H$ بحيث أن $f(H) = G$

ABSTRACT

The aim of this paper is to give a characterization between the external and internal sets, and some relation between the galaxies and monads, according to this paper we obtain the following results:

- A set $G$ is galaxy iff there exists a strictly increasing sequence of internal sets $\{T_n\}_{n \in \mathbb{N}}$ such that $G = \bigcup_{n \in \mathbb{N}} T_n$.

* Assist.Prof.\ College of Computers Sciences and Math.\ University of Mosul
** Assist. Lecturer\ College of Computers Sciences and Math.\ University of Mosul
*** Assist. Lecturer\ College of Computers Sciences and Math.\ University of Mosul

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Also A set $H$ is monad iff there exists a strictly decreasing sequence of internal sets $\{S_n\}_{n \in \mathbb{N}}$ such that $H = \bigcap_{n \in \mathbb{N}} S_n$.

- If $G$ is a galaxy and $H$ is a monad such that $G \subset H$, then there exists an internal set $I$ such that $G \subset I \subset H$.
- A monad is not galaxy.
- If $H$ is a monad and $G$ is a galaxy, the set of all internal functions $f : G \to H$ such that $f(H) = G$ is a monad)

**Keywords:** Galaxy, Monad, Internal, External.

1. **Introduction**

The following definitions and notations are needed throughout this paper:

Every concept concerning sets or elements defined in the classical mathematics is called **standard**.

Any set or formula which does not involve new predicates “standard, infinitesimals, limited, unlimited…etc” is called **internal**, otherwise it is called **external** [3,5].

A real number $x$ is called **unlimited** if and only if $|x| > r$ for all positive standard real numbers $r$; otherwise it is called **limited**.

The notations $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}$ denote respectively the set of real numbers, the set of all unlimited real numbers and the set of all limited real numbers.

A real number $x$ is called **infinitesimal** if $|x| < r$ for all positive standard real numbers $r$.

A real number $x$ is called **appreciable**, if $x$ is limited but not infinitesimal.

Two real numbers $x$ and $y$ are said to be **infinitely close** if and only if $x - y$ is infinitesimal and denoted by $x \simeq y$. [2,4,6,7]

The external set of infinitesimal real numbers is called the **monad of 0** (denoted by $m(0)$). In general, the set of all real numbers, which are infinitely close to a standard real number $a$, is called the **monad of a**, (denoted by $m(a)$)
The set of all limited real numbers is called \textit{principal galaxy}, (denoted by \(G\)).

For any real number \(a\), the set of all real numbers \(x\) such that \(x - a\) limited is called the \textit{galaxy of} \(a\) (denoted by \(G(a)\)).

Let \(a, (a \neq 0)\) and \(x \in \mathbb{R}\), we define the \(\alpha-galaxy(x)\) as follows:
\[
\alpha-galaxy(x) = \{y \in \mathbb{R} : \frac{y-x}{\alpha} \text{ is limited}\}, \text{ and denoted by } \alpha = G(x)
\]

\textbf{Definition 1.1}[3, 6] : A set \(G\) is called galaxy if

(i) \(G\) is an external set.

(ii) there is an internal sequence \(\{A_n\}_{n \in \mathbb{N}}\) of internal sets such that
\[G = \bigcup_{n \in \mathbb{N}} A_n\]

A set \(H\) is called monad if

(i) \(H\) is an external set.

(ii) there is an internal sequence \(\{B_n\}_{n \in \mathbb{N}}\) of internal sets such that
\[H = \bigcap_{n \in \mathbb{N}} B_n\]

\textbf{Theorem 1.2}: (Cauchy Principle) [7]

If \(p\) is any internal property and if \(p(n)\) holds for all standard \(n \in \mathbb{N}\), then there exists an unlimited \(\omega \in \mathbb{N}\) such that \(p(n)\) hold for all \(n \leq \omega\).

\textbf{proposition 1.3} :

(i) If \(\{G_n\}_{n \in \mathbb{N}}\) is a sequence of galaxies then \(\bigcup_{n \in \mathbb{N}} G_n\) is a galaxy.

(ii) If \(\{H_n\}_{n \in \mathbb{N}}\) is a sequence of monads then \(\bigcap_{n \in \mathbb{N}} H_n\) is a monad.

\textbf{Proofs} : follows directly from their definitions.

\textbf{proposition 1.4} :

(i) The image and inverse image of a galaxy under internal mapping are galaxies.
(ii) The image and inverse image of a monad under internal mapping are monads.

**Proofs**: follows directly from their definitions of inverse functions.

Thus we consider the following theorem:

**Theorem 1.5**: 

(i) A set $G$ is galaxy iff there exists a strictly increasing sequence of internal sets $\{T_n\}_{n \in \mathbb{N}}$ such that $G = \bigcup_{n \in \mathbb{N}} T_n$.

(ii) A set $H$ is monad iff there exists a strictly decreasing sequence of internal sets $\{S_n\}_{n \in \mathbb{N}}$ such that $H = \bigcap_{n \in \mathbb{N}} S_n$.

**Proof**: We prove only (i)

Let $G$ be a galaxy and let $\{T_k\}_{k \in \mathbb{N}}$ be an internal sequence of internal sets such that $G = \bigcup_{k \in \mathbb{N}} T_k$. We define a strictly increasing subsequence $\{I_n\}_{n \in \mathbb{N}}$ of the sequence $\{T_k\}_{k \in \mathbb{N}}$. We remark first $T_k \subseteq G$; for all standard $k$, for $G$ is external. So there exists for all standard $k$ a standard natural number $p > k$ such that $T_k \subseteq T_p$. Hence there exists by induction a strictly increasing sequence of standard natural numbers $\{K_n\}_{n \in \mathbb{N}}$ such that $T_{k_n} \subseteq T_{k_{n+1}}$, $n \in \mathbb{N}$. Putting $I_n = T_{k_n}$, we obtain a strictly increasing sequence of internal sets $\{I_n\}_{n \in \mathbb{N}}$ such that $G = \bigcup_{n \in \mathbb{N}} I_n$.

Conversely let $\{I_n\}_{n \in \mathbb{N}}$ be a strictly increasing sequence of internal sets. That is we may Putting $G = \bigcup_{n \in \mathbb{N}} I_n$. By the principle of extension, there exists an internal extension $\{I_n\}_{n \in \mathbb{N}}$ of this sequence, that is we may assume it an increasing. Suppose $G$ is internal set, since $I_n \subseteq G$, for all $n \in \mathbb{N}$, so there exists by Cauchy principle $\omega \in \mathbb{N}$ such that $I_\omega \subseteq G$. Therefore that we may assume it $\bigcup_{n \in \mathbb{N}} I_n \subseteq G$ $\bigcup_{n \in \mathbb{N}} T_n \subseteq G$ which is a contradiction. Hence $G$ is an external set, thus $G$ is a galaxy.

**Remark 1.6**: Let $X$ and $Y$ be two internal sets and $f : X \rightarrow Y$ an internal mapping, and $G_1 \subset X$ and $G_2 \subset Y$ are two galaxies, then

1- If $f \bigg|_{G_1}$ is one to one , then $f(G_1)$ is a galaxy.

2- If $f : X \rightarrow G_2$ is onto, then $f^{-1}(G_2)$ is a galaxy.
Theorem 1.7:

(i) A subset $G$ of an internal set $X$ is a galaxy iff $G$ is the inverse image of $\mathbb{N}$ under an internal mapping from $X$ in to $\mathbb{N}$.

(ii) A subset $H$ of an internal set $X$ is a monad iff $H$ is the inverse image of $\mathbb{N}$ under an internal mapping from $X$ in to $\mathbb{N}$.

Proof:

(i) Let $G \subseteq X$ be a galaxy and let $\{T_n\}_{n \in \mathbb{N}}$ be an internal increasing sequence of internal sets such that $G = \bigcup_{n \in \mathbb{N}} T_n$. We may assume that $\bigcup_{n \in \mathbb{N}} T_n = X$, we define the internal mapping $p: X \to \mathbb{N}$ by $p(x) = \min \{ n \in \mathbb{N} : x \in t_n \}$. Clearly we have $G = p^{-1}(\mathbb{N})$.

Conversely if $p^{-1}(\mathbb{N})$ is a galaxy for every internal mapping $p: X \to \mathbb{N}$ by the proposition (1.4) $p(x) = \min \{..., x \in t_n \}$

(ii) Let $H \subseteq X$ be a monad, putting $G = X - H$, let $p: X \to \mathbb{N}$ be an internal mapping such that $G = p^{-1}(\mathbb{N})$, then we have $H = p^{-1}(\mathbb{N})$.

Conversely if $p^{-1}(\mathbb{N})$ is a monad for every internal mapping $p: X \to \mathbb{N}$ again by the proposition (1.4) we get $H = p^{-1}(\mathbb{N})$.

The converse follow directly from proposition (1, 4)

Proposition 1.8: If $G$ is a galaxy and $H$ is a monad such that $G \subseteq H$, then there exists an internal set $I$ such that $G \subseteq I \subseteq H$.

Proof: Let $\{T_n\}_{n \in \mathbb{N}}$ be an internal increasing sequence of internal sets such that $G = \bigcup_{n \in \mathbb{N}} T_n$ and let $\{K_n\}_{n \in \mathbb{N}}$ be an internal decreasing sequence of internal sets such that $H = \bigcap_{n \in \mathbb{N}} K_n$.

Since $T_n \subseteq K_n$ for all $n \in \mathbb{N}$, there exists by Cauchy principle unlimited real number $\omega$, such that $T_\omega \subseteq K_\omega$ for all natural number $n \leq \omega$ therefore

$$G = \bigcup_{n \in \mathbb{N}} T_n \subseteq \bigcup_{n \leq \omega} T_n = T_\omega \subseteq K_\omega = \bigcap_{n \leq \omega} K_n \subseteq \bigcap_{n \in \mathbb{N}} K_n = H$$

Putting for example $I = T_\omega$, we obtain $G \subseteq I \subseteq H$.

Theorem 1.9: (Fehrel Theorem) [2]

A monad is not galaxy.
Proof: Let $G$ be a galaxy and $H$ be a monad assume that $G \subseteq H$ by Proposition (1.8) we may let $I$ be an internal set such that $G \subseteq I$, $I \subseteq H$ by Cauchy principle an external set is not internal $G \subseteq I$, $I \subseteq H$. Hence $G \neq H$. 

2. Some Application of Fehrel Theorem

Robinson's Lemma 2.1[5]: If $\{a_n\}_{n \in \mathbb{N}}$ is an internal sequence of real numbers such that $a_n \approx 0$, for all $n \in \mathbb{N}$, then there exists an unlimited natural number $\omega \in \mathbb{N}$ such that $a_n \approx 0$, for all $n \leq \omega$.

Proof: Let $b_k$ be the maximum $n \leq k|a_n|$ then also $b_k \approx 0$, for all $n \in \mathbb{N}$

Now by fehrel theorem the galaxy $\mathbb{N}$ is a strictly included in the monad $k$ such that $b_k \approx 0$, the set of all $k$ such that $b_k \approx 0$. So there exists an unlimited $\omega \in \mathbb{N}$ such that $b_\omega \approx 0$, hence $a_n \approx 0$, for all $n \leq \omega$.

3. Functions From a Monad to Galaxy

Now we are able to prove the statement of the following form: The set of internal functions from a monad to galaxy is a galaxy.

Let $G$ be a galaxy and $H$ be a monad. Let $\{A_n\}_{n \in \mathbb{N}}$ be an internal increasing sequence such that $G = \bigcup_{n \in \mathbb{N}} A_n$ and $\{B_n\}_{n \in \mathbb{N}}$ be an internal decreasing sequence such that $H = \bigcap_{n \in \mathbb{N}} B_n$. Then the set $H^G$ of all internal mapping $f : G \to H$ such that $f(G) \subseteq H$ is a monad.

For $f(G) \subseteq H \iff (\forall n \in \mathbb{N})(\forall m \in \mathbb{N})(f(A_n) \subseteq B_m)$

As may be expected $G^H$ is a galaxy we prove this with the help of Fehrel's principle

Proposition 3.1: If $H$ is a monad and $G$ is a galaxy then $G^H$ is a galaxy.

Proof: Let $\{T_k\}_{k \in \mathbb{N}}$ be an internal increasing sequence of internal set such that $G = \bigcup_{n \in \mathbb{N}} T_n$ and let $\{I_n\}_{n \in \mathbb{N}}$ be an internal decreasing sequence of internal set such that $H = \bigcap_{n \in \mathbb{N}} I_n$. We are going to prove that
Clearly if \( f(I_m) = T_n \) for some internal function and \( n, m \in \mathbb{N} \), then \( f(H) \subseteq G \). Also let \( f \) be an internal function such that \( f(H) \subseteq G \), put \( m_n = \min \{ m : f(I_m) \subseteq T_n \} \), for every \( n \in \mathbb{N} \), so \( \{m_n\}_{n \in \mathbb{N}} \) is internal sequence of natural numbers. Now suppose that \( m_n \in \mathbb{N} \), for all \( n \in \mathbb{N} \). Then there exists by fehrel theorem \( \omega \in \overline{\mathbb{N}} \) such that \( m_\omega \in \overline{\mathbb{N}} \) and \( m_n \in \overline{\mathbb{N}} \), so there exists \( x \in I_{m_\omega+1} \) such that \( f(x) \in T_\omega \). Hence \( x \in H \), and \( f(x) \notin G \), contradiction. So there exists \( n \in \mathbb{N} \) such that \( m_n \in \mathbb{N} \). This implies that \( f(I_{m_n}) \subseteq T_n \). Hence \( G^H = \bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} T_n^{I_m} \). So by proposition 1.3 \( G^H \) is a galaxy \( \square \).

**Reference**


