A STUDY OF THE DYNAMICS OF THE FAMILY $\frac{\lambda \sinh^m z}{z^{2m}}$

Hassan Fadhil AL-Husseiny
Department of Mathematics, College of Science, University of Baghdad, Baghdad-Iraq.

Abstract
In this paper, the dynamical behavior of a family of non-critically finite transcendental meromorphic function $f(z) = \frac{\lambda \sinh^m z}{z^{2m}}$, $\lambda > 0$ and $m$ is an even natural number is described. The Julia set of $f(z)$, as the closure of the set of points with orbits escaping to infinity under iteration, is obtained. It is observed that, bifurcation in the dynamics of $f(z)$ occurs at two critical parameter values $\lambda = \lambda_1, \lambda_2$, where $\lambda_1 = \frac{x_1^{2m+1}}{\sinh^m x_1}$ and $\lambda_2 = \frac{x_2^{2m+1}}{\sinh^m x_2}$ with $x_1$ and $x_2$ are the unique positive real roots of the equations $\tanh x = \frac{m}{2m-1}$ and $\tanh x = \frac{m}{2m+1}$ respectively.

Introduction
Let $f(z)$ be a non-constant entire function. Let $z_o = f^o(z_o)$ and $z_n = f(z_{n-1}) = f^o(z_o)$, $n=1,2,...$ where $f^o = f \circ f \circ ... \circ f$ (n-time composition) is the n-th iteration of f. The sequence of iterations at the point $z_o$. The investigation of the (discrete) dynamics of a complex function $f$ is the investigation of its iterations at each point $z_o$ in the extended complex plane $\mathbb{C} \cup \{ \infty \}$ (denoted by $\tilde{\mathbb{C}}$). The
main objects studied in the dynamics of a complex function are its Fatou and Julia sets. The Fatou set (or stable set) of a function $f$, which is denoted by $F(f)$, is defined to be the set of all complex numbers where the family of iterates $\{f^n\}$ of $f$ forms a normal family in the sense of Montel. The Julia set (or chaotic set), denoted by $J(f)$, is the complement of the Fatou set of $f$. In recent years, the dynamics of transcendental functions has been studied. Devaney [1], [2], [3] and Devaney and Durkin [4] studied the dynamics of certain entire transcendental functions such as $\lambda e^z$ and $i \lambda \cos z$. 

The singular values of a function play an important role in determining the dynamics of the function. The dynamics of critically finite meromorphic transcendental functions has been studied for several interesting classes during last two decades [5, 6, 7, 8]. Kapoor and Prasad [9], [10] studied the dynamics of certain non-critically finite entire transcendental functions such as $\lambda (e^z - 1)/z$. In the present work, an effort is made to fill this gap by studying the dynamics of a one parameter family of non-critically finite even transcendental meromorphic functions.

For this purpose, a one parameter family $H= \{ f_z \} = \{ \frac{z \sinh^m z}{z^{2m}}, \lambda > 0, m \in \mathbb{N} \}$ is considered. Bifurcations in the dynamics on real axis for the functions in our family occur at two parameter values. It is observed that the characterization of the Julia set of a function in $H$ as the closure of the set of all its escaping points continues to hold for function in $H$. The Julia set of a function in $H$ is found to contain both real and imaginary axes for certain parameter values.

**Theorem 1.1([6])**

Let $f(z)$ be a transcendental meromorphic function. Suppose $z_o$ lies on an attracting cycle or a parabolic cycle $f(z)$. Then, the orbit of at least one critical value or asymptotic value is attracted to a point in the orbit of $z_o$.

**One parameter family $H$ of non-critically finite functions**

Let $H$ be one parameter family of even transcendental meromorphic functions. The following proposition shows that the functions in the family $H$ are indeed non-critically finite.

**Proposition 2.1** Let $f_z \in H$. Then, the function $f_z(z)$ is non-critically finite.

**Proof** The derivative of the function $f_z(z)$ for $z \neq 0$ is given by

$$f'_z = \lambda \frac{m \sinh^{m-1} z (z \cosh z - 2 \sinh z)}{z^{2m+1}}$$

The critical points of the function $f_z(z)$ are solutions of the equation $f'_z(z) = 0$. and these solutions are $z = k \pi i$, where $k$ is a non-zero integer and solutions of the equation

$$z \cosh z - 2 \sinh z = 0$$

are the critical points of $f_z(z)$. The solutions of equation (2.1) are the same as the solutions of the equation

$$\tanh z = \frac{z}{2}$$

have a solution $z_0$, if and only if the equation $\tan w = \frac{w}{2}$ has a solution $iz_0$. Now equating the real and imaginary parts of the equation $\tan w = \frac{w}{2}$, for a non-zero $z = x+iy$,

$$\frac{\sin 2x}{\cos 2x + \cosh 2y} = \frac{x}{2}$$

and

$$\frac{\sinh 2y}{\cosh 2x + \cosh 2y} = \frac{y}{2}.$$ This implies to

$$\frac{\sin 2x}{x} = \frac{\sinh 2y}{y} \quad (2.2)$$

It is easily to show that, for $x, y \neq 0$, the

$$\left| \frac{\sin 2x}{x} \right| < 2 \quad \text{and} \quad \left| \frac{\sinh 2y}{y} \right| > 2$$

so that (2.2) is not possible in this case. Therefore, at least one of $x, y$ must be zero. Hence (2.2) has only real or purely imaginary roots. If $x = 0$, then $\tan w = \frac{w}{2}$ implies that $\tanh y = \frac{1}{2} y$ and this equation has two zeros. Therefore, $\tan w = \frac{w}{2}$ has two purely imaginary solutions. If $y = 0$, then $\tan w = \frac{w}{2}$ implies that $\tan x = \frac{1}{2} x$ and this equation has infinitely many purely imaginary solutions. Thus, the function $f_z(z)$ has two real and
infinitely many imaginary critical points. To find the critical values of the function $f_\lambda(z)$, we note that $f_\lambda(k,\text{i}) = 0$, where $k$ is a non-zero integer. Let $\{i, y_L\}_{L=-\infty}^{\infty}$ be the critical points of $f_\lambda(z)$ other than the critical points $k$, for $k = \pm 1, \pm 2, \ldots$ since $f_\lambda(i, y_L) = \lambda m L L \sin x$ and the values $m L L \sin x$ are real and distinct, it follows that the values in the set $\{f_\lambda(i, y_L)\}_{L=-\infty}^{\infty}$ are real and distinct. Therefore, the function $f_\lambda(z)$ possesses infinitely many real critical values.

Fixed points and their nature for functions in $H$

In this section, we find the fixed points of the function $f_\lambda(x) = \lambda x x x x x x x x$ and describe their nature. Let

$$\Phi(x) = \frac{x^{m+1}}{\sinh^m x}$$

The function $\Phi(x)$ has the following properties:

1. $\Phi(x)$ is continuous in $\mathbb{R}$.
2. $\Phi(x)$ is positive in $(0, \infty)$, is negative in $(-\infty,0)$.
3. $\Phi(x) \to 0$ as $x \to -\infty$ and $\Phi(x) \to 0$ as $x \to \infty$.

Further,

4. $\Phi'(x)$ is continuous in $\mathbb{R}$. Since

$$\Phi'(x) = \frac{(2m+1)x^m \sinh x - mx^{m+1} \cosh x}{\sinh^{m+1} x},$$

it follows easily that $\Phi'(0) = \lim_{x \to 0} \Phi(x)$, so that $\Phi'(x)$ is continuous in $\mathbb{R}$.

5. $\Phi(x)$ has a unique positive real zero at $x = x_2$, where $x_2$ is a real positive solution of $\tanh x = \frac{mx}{2m+1}$; since $\Phi(x) = 0$ gives $\tanh x = \frac{mx}{2m+1}$ and by Newton-Raphson's method $x_2$ is a real positive solution of $\Psi(x) = \tanh x - \frac{mx}{2m+1} = 0$. (Fig.1 (a)), the property (5) follows.

6. $\Phi(x)$ is strictly increasing in $(0, x_2)$, is strictly decreasing in $(x_2, \infty)$ and has a maximum at $x = x_2$, where $x_2$ is a real positive solution of $\tanh x = \frac{mx}{2m+1}$.

By property (5), $\Phi'(x_2) = 0$, where $x_2$ is a real positive solution of $\tanh x = \frac{mx}{2m+1}$. $\Phi''(x) = \frac{\sinh x [(2m+1)x^m \cosh x + 2m(2m+1)x^{m-1} \sinh x]}{\sinh^{m+2} x} - \frac{(m+1) \cosh x [(2m+1)x^m \sinh x - mx^{m+1} \cosh x]}{\sinh^{m+2} x}$

Since $\Phi''(x_2) < 0$. Therefore, the function $\Phi(x)$ has exactly one maxima in $(0, \infty)$ at $x = x_2$. It therefore follows by property (3) that $\Phi(x)$ decreases to 0 in $(x_2, \infty)$ and increases in $(0, x_2)$. The graph of $\Phi(x)$ therefore is as shown in Fig.1 (b).
Throughout the sequel, we denote
\[ \lambda_2 = \Phi(x_2) \] (3.2)
where \( x_2 \) is unique positive real solution of the equation \( \tanh x = \frac{mx}{2m+1} \).

The following proposition gives the number and locations of real fixed points of the function \( f_\lambda(x) \) for \( \lambda > 0 \):

**Proposition 3.1**

Let \( f_{\lambda} \in \mathbb{H} \). Then the locations of real fixed points of the function \( f_{\lambda}(x) = \frac{\sinh^m x}{x^{2m}} \) are given by the following:

1- For \( 0 < \lambda < \lambda_2 \), \( f_{\lambda}(x) \) has exactly one fixed point in each of the intervals \((0, x_2)\) and \((x_2, \infty)\), where \( x_2 \) is a positive real solution of the equation \( \tanh x = \frac{mx}{2m+1} \).

2- For \( \lambda = \lambda_2 \), the only fixed point of \( f_{\lambda}(x) \) is at \( x = x_2 \) where \( x_2 \) is as in (1).

3- For \( \lambda > \lambda_2 \), \( f_{\lambda}(x) \) has no fixed points.

**Proof.** The fixed points of the function \( f_{\lambda}(x) \) are the solutions of the equation \( \lambda = \Phi(x) \), where \( \Phi(x) \) is given by (3.1). We have the following cases:

1- \( 0 < \lambda < \lambda_2 \)

Since \( \Phi(x_2) = \lambda_2 \) and \( \lambda < \lambda_2 \), in view of properties (1), (3) and (6) of the function \( \Phi(x) \), the line \( u = \lambda \) intersects the graph of \( \Phi(x) \) at exactly two points. Using properties (2), (3), and (6), it follows in view of \( \Phi(x_2) = \lambda_2 \) that one of the solutions of \( \Phi(x) = \lambda \) for \( 0 < \lambda < \lambda_2 \) lies in the interval \((0, x_2)\). Similarly, since by property (6) \( \Phi(x) \) is decreasing in the interval \((x_2, \infty)\) and \( \Phi(x_2) = \lambda_2 \), the other solution of \( \Phi(x) = \lambda \) for \( 0 < \lambda < \lambda_2 \) lies in the interval \((x_2, \infty)\). Thus, \( f_{\lambda}(x) \) has two real fixed points lying in the intervals \((0, x_2)\) and \((x_2, \infty)\).

2- \( \lambda = \lambda_2 \)

The function \( \Phi(x) \) has exactly one maxima at \( x = x_2 \) and the maximum value of \( \Phi(x) \) is \( \Phi(x_2) = \lambda_2 \). The line \( u = \lambda_2 \) intersects the graph of \( \Phi(x) \) at exactly one point at \( x = x_2 \). Therefore, the equation \( \Phi(x) = \lambda_2 \) has exactly one solution at \( x = x_2 \). Thus, \( f_{\lambda}(x) \) has only one real fixed point at \( x = x_2 \) for \( \lambda = \lambda_2 \).

3- \( \lambda > \lambda_2 \)

By property (6), the maximum value of \( \Phi(x) \) is \( \Phi(x_2) = \lambda_2 \), therefore, for \( \lambda > \lambda_2 \), the line \( u = \lambda \) does not intersect the graph of \( \Phi(x) \). Consequently, the equation \( \Phi(x) = \lambda \) has no solution for \( \lambda > \lambda_2 \). Thus \( f_{\lambda}(x) \) has no fixed point for \( \lambda > \lambda_2 \).

Now, Let
\[ \lambda_1 = \Phi(x_1) \] (3.3)
where \( x_1 \) is a positive solution of the equation \( \tanh x = \frac{mx}{2m+1} \). The fixed points of the function \( f_{\lambda}(x) \) found in proposition (3.1) are denoted by \( r_1 \in (0, x_1) \), \( r_2 \in (x_3, \infty) \), \( a_1 \in (x_1, x_2) \) and \( r_3 \in (x_2, x_3) \), where \( x_2 \) is a positive solution of \( \tanh x = \frac{mx}{2m+1} \) and \( x_1, x_3 \) be solutions of \( \lambda_1 \) = \( \Phi(x) \) lying in the intervals \((0, x_2)\) and \((x_2, \infty)\) respectively.

The nature of these fixed points of the function \( f_{\lambda}(x) \) for different values of parameter \( \lambda \) is described in the following theorem:

**Theorem 3.1**

Let \( f_{\lambda}(x) = \frac{\sinh^m x}{x^{2m}} \) for \( x \in \mathbb{R}\{0\} \) and \( x_1, x_3 \) be solutions of \( \lambda_1 = \Phi(x) \) lying in the intervals \((0, x_2)\) and \((x_2, \infty)\) respectively, where \( x_2 \) is a positive solution of the equation \( \tanh x = \frac{mx}{2m+1} \).

1- If \( 0 < \lambda < \lambda_1 \), then the fixed points \( r_1 \in (0, x_1) \) of \( f_{\lambda}(x) \) and \( r_2 \in (x_3, \infty) \) of \( f_{\lambda}(x) \) are repelling.

2- If \( \lambda = \lambda_1 \), then the fixed point \( x_1 \) of \( f_{\lambda}(x) \) is indifferent and the fixed point \( x_3 \) of \( f_{\lambda}(x) \) is repelling.

3- If \( \lambda_1 < \lambda < \lambda_2 \), then the fixed point \( a_1 \in (x_1, x_2) \) of \( f_{\lambda}(x) \) is attracting and the fixed point \( r_2 \in (x_2, x_3) \) of \( f_{\lambda}(x) \) is repelling.

4- If \( \lambda = \lambda_2 \), then the fixed point \( x_2 \) of \( f_{\lambda}(x) \) is indifferent.

**Proof:** since the derivative of the function \( f_{\lambda}(x) \) is given by
\[ f'_{\lambda}(x) = \lambda \frac{[m(x\sinh^{m-1} x \cosh x - 2\sinh^m x)]}{x^{2m+1}} \]
and the fixed points of the function $f_M(x)$ are solution of $\lambda = \frac{x^{2n+1}}{\sinh^n x}$, it follows that the multiplier $f_M'(x_f)$ of the fixed point $x_f$ is given by
\[
|f_M'(x_f)| = m|\text{coth}(x_f) - 2| \quad (3.4)
\]

Let $G(x) = \begin{cases} m(x \cot x - 2) & \text{for } x \neq 0 \\ -2 & \text{for } x = 0 \end{cases}$

The function $G(x)$ is differentiable and its derivative is given by
\[
G'(x) = \begin{cases} m(\cot x - x \csc h^2 x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}
\]

Since, $G'(x) \neq 0$, $G'(0) = 0$ and $G''(0) > 0$, the function $G(x)$ has exactly one minima at $x = 0$ and the minimum value is (-2). Since $G(x) > 0$ for $x \in (0, \infty)$ and $G'(x) < 0$ for $x \in (-\infty, 0)$, thus $G(x)$ increases for $(0, \infty)$ and decreases for $(-\infty, 0)$. Therefore, it follows that the function $|G(x)|$ (Fig. 2) satisfies
\[
|G(x)| = \begin{cases} <1 & \text{for } x \in (-x_2, -x_1) \cup (x_1, x_2) \\ 1 & \text{for } x = \pm x_1, \pm x_2 \\ >1 & \text{for } x \in (-\infty, -x_2) \cup (-x_1, 0) \cup (0, x_1) \cup (x_2, \infty) \end{cases}
\]

1- $0 < \lambda < \lambda_1$

Since the fixed point $r_1 \in (0, x_1)$, by inequality (3.7), $|f_M'(r_1)| > 1$. It therefore follows that $r_1$ is a repelling fixed point of $f_M(x)$. Similarly, since the fixed point $r_2 \in (x_2, \infty)$, by inequality (3.7), $|f_M'(r_2)| > 1$. Consequently, $r_2$ is a repelling fixed point of $f_M(x)$.

2- $\lambda = \lambda_1$

By equation (3.6), $|f_M'(x_1)| = 1$, therefore, $x = x_1$ is an indifferent fixed point of $f_M(x)$. Further, since $x_1 > x_2 > x_1$, it follows that $x_1 \in (x_2, \infty)$. By inequality (3.7), $|f_M'(r_1)| > 1$. It therefore follows that $x_3$ is a repelling fixed point of $f_M(x)$.

3- $\lambda_1 < \lambda < \lambda_2$

Since the fixed point $a_k \in (x_1, x_2)$, by inequality (3.5), $|f_M'(a_k)| < 1$. Thus $a_k$ is an attracting fixed point of $f_M(x)$. Further, since the fixed point $r_3 \in (x_2, x_3)$, by inequality (3.7) gives that $|f_M'(r_3)| > 1$. It therefore follows that $r_3$ is a repelling fixed point of $f_M(x)$.

4- $\lambda = \lambda_2$

By equation (3.6), $|f_M'(x_2)| = 1$. Consequently, $x = x_2$ is an indifferent fixed point of $f_M(x)$.

**Bifurcations of the family H on $\mathbb{R} \setminus \{0\}$**

In this section, the dynamics of functions $f_M \in H$ on the real line is described. It is proved in the following theorem that there exist parameter values $\lambda_1, \lambda_2 > 0$ such that bifurcations in the dynamics of the function $f_M(x), x \in \mathbb{R}\setminus T_0$ occur at $\lambda = \lambda_1$ and $\lambda = \lambda_2$, where $T_0$ is the set of the points that are backward orbits of the pole 0 of the function $f_M(x)$.

**Theorem 4.1.** Let $f_M(x) = \lambda \frac{\sinh^n x}{x^{2n}}$ for $x \in \mathbb{R} \setminus \{0\}$.

a. If $0 < \lambda < \lambda_1$, $f_M^n(x) \rightarrow \infty$ as $n \rightarrow \infty$ for $x \in \{(-\infty, -2) \cup (-\alpha_1, 0) \cup (0, \alpha_1) \cup (2, \infty) \} \setminus T_0$ and the orbits $\{f_M^n(x)\}$ are chaotic for $x \in \{(-2, -r_1)$

\[f_M'(x_f) = \begin{cases} -1 & \text{for } x = \pm x_1, \pm x_2 \\ >1 & \text{for } x \in (-\infty, -x_2) \cup (-x_1, 0) \cup (0, x_1) \cup (x_2, \infty) \end{cases}
\]

![Figure 2: Graph of $|G(x)|$](image-url)

Consequently, by (3.4), we get that the multiplier $f_M'(x_f)$ of the fixed point $x_f$ satisfies
The function \( f_\lambda^a(x) \) is continuously differentiable for \( x \rightarrow \infty \) and \( f_\lambda^a(x) > 0 \) for sufficiently small \( \delta > 0 \) and \( t_\lambda(x) < 0 \) in \((r_2-\delta_2, r_2)\) since \( f_\lambda(x) \neq 0 \) in \((0, r_1) \cup (r_1, r_2) \cup (r_2, \infty)\) it now follows that \( t_\lambda(x) > 0 \) in \((0, r_1) \cup (r_1, r_2) \cup (r_2, \infty)\) and \( t_\lambda(x) < 0 \) in \((r_1, r_2)\). Thus,

\[
\begin{align*}
t_\lambda(x) &> 0 & & \text{for } x \in (0, r_1) \cup (r_2, \infty) \\
t_\lambda(x) &< 0 & & \text{for } x \in (r_1, r_2) \\
\end{align*}
\]

(4.1)

The dynamics of the function \( f_\lambda(x) \) is now described by the following cases:

**Case-i** \( x \in \left[\left(-r_2, -r_1\right) \cup \left(-a_1, 0\right) \cup (0, a_1) \cup (r_2, \infty)\right] \): By (4.1), it follows that, for \( x \in \left(r_2, \infty\right) \), \( f_\lambda(x) > x \). Since the function \( f_\lambda(x) \) is increasing for \( x \in \left(r_2, \infty\right) \), \( f_\lambda^n(x) \rightarrow \infty \) as \( n \rightarrow \infty \). Further, since \( f(a_1) = r_2 \) and \( f_\lambda^1(x) \) is decreasing in \((0, a_1)\), the function \( f_\lambda(x) \) maps the interval \((0, a_1)\) into \((r_2, \infty)\). Now, using the above arguments, we get \( f_\lambda^n(x) \rightarrow \infty \) as \( n \rightarrow \infty \) for \( x \in (0, a_1) \). Next, since \( f_\lambda(x) \) is an even function, using the above arguments again, we get \( f_\lambda^n(x) \rightarrow \infty \) as \( n \rightarrow \infty \) for \( x \in \left[\left(-r_2, -r_1\right) \cup (-a_1, 0)\right] \).

**Case-ii** \( x \in \left[\left(-r_2, -r_1\right) \cup (-a_1, r_1) \cup (r_1, r_2) \right] \): Since there is no attractor to attract the system dynamics, the dynamical system will keep moving indefinitely. Therefore, orbits of \( f_\lambda(x) \) are chaotic for \( x \in \left(a_1, r_1\right) \cup \left(r_1, r_2\right) \). Further, since \( f_\lambda(x) \) is an even function, using the above arguments again, orbits of the function \( f_\lambda(x) \) are chaotic for \( x \in \left[\left(-r_2, -r_1\right) \cup (-a_1, a_1)\right] \).

**b-** If \( \lambda = \lambda_1 \), by Theorem 3.1, the function \( f_\lambda(x) \) has an indifferent fixed point \( x_1 \) and a repelling fixed point \( x_3 \). Since \( t_\lambda'(x_1) < 0 \) and in a neighborhood of \( x_1 \) the function \( t_\lambda'(x) \) is continuous, \( t_\lambda'(x) < 0 \) in some neighborhood of \( x_1 \). Therefore, \( t_\lambda(x) \) is decreasing in a neighborhood of \( x_1 \). By the continuity of \( t_\lambda(x) \), for sufficiently small \( \delta_1 > 0 \), \( t_\lambda(x) < 0 \) in \((x_1-\delta_1, x_1)\) and \( t_\lambda(x) > 0 \) in \((x_1, x_1+\delta_1)\) and \( t_\lambda(x) < 0 \) in \((r_2, r_2+\delta_2)\) and \( t_\lambda(x) > 0 \) in \((r_2-\delta_2, r_2)\).
since \( t'_2(x_3) > 0 \) and in a neighborhood of \( x_1 \) the function \( t'_2(x) \) is continuous, \( t'_2(x) > 0 \) in some neighborhood of \( x_3 \). Therefore, \( t'_2(x) \) is increasing in a neighborhood of \( x_1 \). By the continuity of \( t'_2(x) \), for sufficiently small \( \delta_2 > 0 \), \( t'_2(x) > 0 \) in \((x_3, x_3 + \delta_2)\) and \( t'_2(x) < 0 \) in \((x_3 - \delta_2, x_3)\). Since \( t'_2(x) \neq 0 \) in \((0, x_1) \cup (x_1, x_3) \cup (x_3, \infty)\), it now follows that \( t'_2(x) > 0 \) in \((0, x_1) \cup (x_3, \infty)\) and \( t'_2(x) < 0 \) in \((x_1, x_3)\). Thus,

\[
\begin{align*}
t'_2(x) &> 0, & x \in (0, x_1) \cup (x_3, \infty) \\
t'_2(x) &< 0, & x \in (x_1, x_3)
\end{align*}
\tag{4.2}
\]

The dynamics of the function \( f'_2(x) \) is now described by the following cases:

**Case-i** (\( x \in [(-x_3, -a_2) \cup (a_2, x_3)] \cup (\infty, \infty) \)):

By \( (4.2) \), it follows that \( f'_2(x) < 1 \) for \( x \in (a_2, x_3) \), \( f'_2(x) = 1 \) and \( f'_2(x) > 1 \) for \( x < x_3 \), it follows that, using Mean Value Theorem, 

\[
|f'(x) - x_1| < |x - x_1| \quad \text{for} \quad x \in (a_2, x_3).
\]

Therefore, \( f'_2(x) \to x_1 \) as \( n \to \infty \) for \( x \in (a_2, x_3) \).

Further, since \( f'_2(x) \) is an even function, using the above arguments again, \( f''_2(x) \to x_1 \) as \( n \to \infty \) for \( x \in (x_1, x_3) \).

**Case-ii** (\( x \in [(-\infty, -x_3) \cup (-a_2, 0) \cup (0, -a_2) \cup (x_3, \infty)] \cup (\infty, \infty) \)):

By \( (4.2) \), it follows that, for \( x \in (x_3, \infty) \), \( f'_2(x) > x \). Since \( f'_2(x) \) is increasing for \( x \in (x_3, \infty) \), so that \( f''_2(x) \to \infty \) as \( n \to \infty \). Further, since \( f'_2(x) > x_3 \) and \( f'_2(x) \) is decreasing in \((0, a_2)\), so that \( f'_2(x) \) maps the interval \((0, a_2)\) into \((x_3, \infty)\). Now, using the above arguments, we get \( f''_2(x) \to \infty \) as \( n \to \infty \) for \( x \in (0, a_2) \).

Next, since \( f'_2(x) \) is an even function, using the above arguments again, \( f''_2(x) \to \infty \) as \( n \to \infty \) for \( x \in (-\infty, -x_3) \cup (\infty, \infty) \).

\[c.\] If \( \lambda_1 < \lambda < \lambda_2 \), by Theorem 3.1, the function \( f'_2(x) \) has an attracting fixed point \( a_0 \) and a repelling fixed point \( r_1 \). Since, \( f'_2(a_0) < 0 \) and in a neighborhood of \( a_0 \) the function \( f'_2(x) \) is continuous, \( f'_2(x) < 0 \) in some neighborhood of \( a_0 \). Therefore, \( f'_2(x) \) is decreasing in a neighborhood of \( a_0 \). By the continuity of \( f'_2(x) \), for sufficiently small \( \delta_1 > 0 \), \( f'_2(x) > 0 \) in \((a_0 - \delta_1, a_0)\) and \( f'_2(x) < 0 \) in \((a_0, a_0 + \delta_1)\). Further, since \( f'_2(r_1) > 0 \) and in a neighborhood of \( r_1 \), the function \( f'_2(x) \) is continuous, \( f'_2(x) > 0 \) in some neighborhood of \( r_1 \). Therefore, \( f'_2(x) \) is increasing in a neighborhood of \( r_1 \). By the continuity of \( f'_2(x) \), for sufficiently small \( \delta_3 > 0 \), \( f'_2(x) > 0 \) in \((r_1, r_1 + \delta_3)\) and \( f'_2(x) < 0 \) in \((r_1 - \delta_3, r_1)\).

Since \( f'_2(x) \neq 0 \) in \((0, a_2) \cup (a_2, r_1)\), it now follows that \( f'_2(x) > 0 \) in \((0, a_2) \cup (a_2, r_1) \cup (r_1, \infty)\) and \( f'_2(x) < 0 \) in \((a_2, r_1)\). Thus,

\[
\begin{align*}
f'_2(x) &> 0, & x \in (0, a_2) \cup (r_1, \infty) \\
f'_2(x) &< 0, & x \in (a_2, r_1)
\end{align*}
\tag{4.3}
\]

The dynamics of the function \( f'_2(x) \) is now described by the following cases:

**Case-i** (\( x \in [(-r_2, -a_3) \cup (a_3, r_2)] \cup (\infty, \infty) \)):

Let \( x_0 \) be a positive solution of the equation \( f'(x) = 0 \). Thus \( f''(x_0) > 0 \), hence \( f'_2(x) \) has a minimum at \( x_0 \). Since \( f'_2(a_0) < 1, f'_2(x_0) = 0, \)

\( f'_2(x) \) is increasing for \( x > 0 \) and \( a_0 < x_0 \), there exists a point \( b \in \{x_0, r_1\} \) such that \( f'_2(b) < 1 \) for all \( c \in [a_0, b] \supseteq [a_0, x_0] \). Using Mean Value Theorem, it follows that \( f'_2(x) < f'_2(a_0) < 1 \) for \( x \in [a_0, b] \).

Consequently, \( f''_2(x) \to a_0 \) as \( n \to \infty \) for \( x \in [a_0, b] \). For each \( x \in [b, r_2] \) the forward orbits contain a point from \([a_0, b] \). Therefore, same as above, \( f''_2(x) \to a_0 \) as \( n \to \infty \) for all \( x \in [b, r_2] \).

Hence \( f''_2(x) \to a_0 \) as \( n \to \infty \) for \( x \in [a_0, r_2] \).

Again, since \( f'_2(a_3) = r_2 \) and \( f'_2(x) \) is decreasing in the interval \((a_3, a_2)\), \( f'_2(x) \) maps the interval \((a_3, a_2)\) into \([a_2, r_2] \).

Therefore, using the above arguments again, \( f''_2(x) \to a_0 \) as \( n \to \infty \) for \( x \in [a_0, r_2] \).
Further, since \( f_1'(x) \) is an even function, using the above arguments again, \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in (a_3, a_4]\). Thus, \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in (\alpha_3, \alpha_4) \). Therefore, \( f_1^n(x) \rightarrow a_3 \) as \( \lambda \rightarrow \alpha_3 \) for \( x \in (\alpha_3, \alpha_4) \).

**Case-II** 

\( (x, x^2, \lambda) \) has an indifferent fixed point at \( x = 2 \). Since \( \lambda \rightarrow \alpha_3 \), \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in (\alpha_3, \alpha_4) \). Further, since \( f_1(x) = a_3 \) and \( f_1(x) \) is decreasing from \( 0 < x < a_3 \), \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in (0, a_3) \). Therefore, using the above arguments again, \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in (\alpha_3, \alpha_4) \).

By (4.4), it follows that, for \( x > 0 \), \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in (\alpha_3, \alpha_4) \). Further, since \( f_1(x) = a_3 \) and \( f_1(x) \) is decreasing from \( 0 < x < a_3 \), \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in (0, a_3) \). Further, since \( f_1(x) \) is an even function, \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in [-\alpha_3, \alpha_3] \). Thus, \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in [-\alpha_3, \alpha_3] \). Further, since \( f_1(x) \) is an even function, \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in [-\alpha_3, \alpha_3] \). Thus, \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in [-\alpha_3, \alpha_3] \).

**Case-II** 

\( (x, x^2, \lambda) \) has a fixed point at \( x = 2 \). Since \( \lambda \rightarrow \alpha_3 \), \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in (\alpha_3, \alpha_4) \). Further, since \( f_1(x) = a_3 \) and \( f_1(x) \) is decreasing from \( 0 < x < a_3 \), \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in (0, a_3) \). Further, since \( f_1(x) \) is an even function, \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in [-\alpha_3, \alpha_3] \). Thus, \( f_1^n(x) \rightarrow a_3 \) as \( n \rightarrow \infty \) for \( x \in [-\alpha_3, \alpha_3] \).

It follows by Theorem 4.1 that bifurcation in the dynamics of the function \( f_1^n(x) \) for all \( x \in \mathbb{R} \{0\} \) occur at the two critical parameter values \( \lambda = \lambda_1, \lambda_2 \), where \( \lambda_1 = \frac{x_1^{2m+1}}{\sinh^{m} x_1} \) and \( \lambda_2 = \frac{x_2^{2m+1}}{\sinh^{m} x_2} \) such that \( x_1 \) and \( x_2 \) are the unique positive real roots of the equations \( \tanh x = \frac{mx}{2m-1} \) and \( \tanh x = \frac{mx}{2m+1} \) respectively.

**Dynamics on \( \mathbb{C} \)**

The dynamics of functions from the one-parameter family \( \mathbb{H} \) is indicated by describing the dynamics of \( f_1^n \in \mathbb{H} \) for \( z \in \mathbb{C} \) in the present section. This includes the study of Julia set of \( f_1 \) in the extended complex plane \( \mathbb{C} \) for different values of \( \lambda \in \mathbb{R} \). If the singular values of transcendental function is bounded, then the Julia set is a closure of escaping points of function under iterations [11]; i.e., \( J(f) = \{ z \in \mathbb{C} : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty \text{ and } f^n(z) \neq \infty \} \).
**Proposition 5.1** Let $f_\lambda \in H$ and $0 < \lambda < \lambda_1$. Then the Julia set $J(f_\lambda)$ contains both real and imaginary axes.

**Proof** By Theorem 4.1(a), $f_\lambda^n(x) \to \infty$ for $x \in \{(-\infty, -r_2) \cup (-r_1, 0) \cup (0, a_3) \cup (r_2, \infty)\} \mathcal{T}_\lambda$ and the orbits $\{f_\lambda^n(x)\}$ are chaotic for $x \in \{(-r_1, -r_2) \cup (r_1, r_2)\} \mathcal{T}_\lambda$, it follows that $R \mathcal{T}_\lambda \subset J(f)$. In addition, since the poles and their preimages are contained in the Julia set, thus $R \subset J(f)$. Now, we have $f_\lambda$ maps the real interval into $iR$. Therefore, the forward orbits of all singular values tend to $\infty$. Therefore, $J(f)$ contains both real and imaginary axes.

**Proposition 5.2** Let $f_\lambda \in H$ and $\lambda = \lambda_1$. Then

1) The Fatou set $F(f)$ contains a unique parabolic domain.

2) The Julia set contains the intervals $(-\infty, -x_2)$, $(x_2, \infty)$.

The sequence of iterates $\{f_\lambda^n(z)\}$ tends to $a_\lambda$ as $n \to \infty$ so that the sequence of iterates $\{f_\lambda^n(z)\}$ forms normal family at $z$. Consequently, $z \in F(f)$. Thus, $A(a_\lambda) \subset F(f)$. Further, by Theorem 4.1(c), it follows that the forward orbits of all singular values either tend to $a_\lambda$ or tend to $\infty$. Therefore, by Theorem 1.1, $F(f)$ does not contain the basin of attractions other than $A(a_\lambda)$. That $F(f)$ does not contain any parabolic domains follows similarly using Theorem 1.1.

**Proposition 5.3** Let $f_\lambda \in H$ and $\lambda_1 < \lambda < \lambda_2$. Then

1) The Fatou set $F(f)$ does not contain any basin or parabolic domain except the basin of attraction of the real attracting fixed point $a_\lambda$ of $f_\lambda(x)$.

2) The intervals $\{(-r_2, a_3) \cup (a_3, r_2)\} \mathcal{T}_\lambda$ are contained in $F(f)$ and the intervals $\{(-\infty, -r_1), (x_2, \infty)\}$ are contained in $J(f)$.

**Proof**

1) By Theorem 3.1(3), $f_\lambda(z)$ has a real attracting fixed point $a_\lambda$. Let $A(a_\lambda) = \{z \in \mathbb{C} ; f_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}$ be the basin of attraction of the attracting fixed point $a_\lambda$. For any point $z \in A(a_\lambda)$, the sequence of iterates $\{f_\lambda^n(z)\}$ tends to $a_\lambda$ as $n \to \infty$ so that $A(a_\lambda) \subset F(f)$. Further, by Theorem 4.1(c), it follows that the forward orbits of all singular values either tend to $a_\lambda$ or tend to $\infty$. Therefore, by Theorem 1.1, $F(f)$ does not contain the basin of attractions other than $A(a_\lambda)$. That $F(f)$ does not contain any parabolic domains follows similarly using Theorem 1.1.
The proof of proposition is analogous to that of proposition 5.2 for the case \( \lambda = \lambda_1 \) and is hence omitted.

**Proposition 5.5** Let \( f_\lambda \in H \) and \( \lambda > \lambda_2 \). Then, the Julia set \( J(f) \) contains both real and imaginary axes.

**Proof** By Theorem 4.1(e), \( f_\lambda^n(x) \to \infty \) for all \( x \in \mathbb{R} \setminus \mathbb{T}_0 \), it follows that \( \mathbb{R} \setminus \mathbb{T}_0 \subset J(f) \). Since \( f_\lambda^n(x) \) maps imaginary axis on real axis and \( f_\lambda^n(x) \to \infty \) for all \( x \in \mathbb{R} \setminus \mathbb{T}_0 \), it gives that \( i\mathbb{R} \setminus i\mathbb{T}_0 \subset J(f) \). Since the asymptotic value 0 is also a pole of \( f_\lambda(z) \) \( f_\lambda(z) \cdot 0 \in J(f) \) and since preimages of pole are contained in Julia set, \( \mathbb{T}_0 \subset J(f) \). Therefore, \( J(f) \) contains both real and imaginary axes.

**Proposition 5.6** For \( \lambda > \lambda_2 \), Fatou set cannot have any basin of attraction, parabolic domain.

**Proof** Since \( f_\lambda^n(x) \to \infty \) for all \( x \in \mathbb{R} \setminus \mathbb{T}_0 \), the forward orbit of critical values on real axis tend to \( \infty \). Further, the asymptotic value 0 is also a pole of \( f_\lambda(z) \) so that orbit of 0 termintes. Therefore, Fatou set cannot have any basin of attraction, parabolic domain.

**References**
