Construction of Complete (k,n)-arcs in the Projective Plane
PG(2,11) Over Galois Field GF(11), 3 ≤ n ≤ 11

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Abstract
The purpose of this work is to construct complete (k,n)-arcs in the projective 2-space PG(2,q) over Galois field GF(11) by adding some points of index zero to complete (k,n–1)-arcs 3 ≤ n ≤ 11.

A (k,n)-arcs is a set of k points no n + 1 of which are collinear.
A (k,n)-arcs is complete if it is not contained in a (k + 1,n)-arcs.

Introduction
Mayssa 2004 (4), constructed of complete (k,n)-arcs in PG(2,17) and Sawsan 2001 (6), showed the classification and construction of (k,n)-arcs from (k,m)-arcs in PG(2,q) m < n. And Ban, (8) showed the classification and construction of (k,4)-arc, k = 17, 18,…, 34, in PG(2,11).

This paper is divided into two sections, section one consists of proving basic, theorems and giving some definitions of projective plane, (k,n)-arcs, maximal and complete arcs…ets. Section two consists of the projective plane of order eleven. The construction of complete (k,2)-arcs call it c1, c2, c3, …, c9 and the construction of complete (k,n)-arcs from complete (k,n – 1)-arcs in PG(2,11), where n = 3, 4, …, 9, 10 gave the points P and lines L in PG(2,11) are determined in the table (1,1).

Section One
1.1 Definition "Projective Plane" (1)
A projective plane PG(2,q) over Galois field GF(q) is a two-dimensional projective space, which consists of points and lines with incidence relation between them. In PG(2,q) there are q^2 + q + 1 points, and q^2 + q + 1 lines, every line contains 1 + q points and every point is on 1 + q lines, all these points in PG(2,q) have the form of a triple (a1,a2,a3) where a1, a2, a3 ∈ GF(q); such that (a1,a2,a3) ≠ (0,0,0). Two points (a1,a2,a3) and (b1,b2,b3) represent the same point if there exists λ ∈ GF(q) \{0\}, such that (b1,b2,b3) = λ(a1,a2,a3).

There exists one point of the form (1,0,0). There exists q points of the form (x,1,0). There exists q^2 points of the form (xy,1), similarly for the lines.
A point p(x1,x2,x3) is incident with the line L[a1,a2,a3] if and only if a1x1 + a2x2 + a3x3 = 0, i.e.
A point represented by (x1,x2,x3) is incident with the line represented by \( \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \), if
(\( x_1 x_2 x_3 \)) \( \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \) = 0 ⇒ a1x1 + a2x2 + a3x3 = 0.

The projective plane PG(2,q) satisfying the following axioms:
1. Any two distinct lines intersected in a unique point.
2. Any two distinct points are contained in a unique line.
3. There exists at least four points such that no three of them are collinear.
1.2 Definition (1)
Two lines \([a_1,a_2,a_3]\) and \([b_1,b_2,b_3]\) represent the same line if there exists \(\lambda \in GF(q) \setminus \{0\}\), such that \([b_1,b_2,b_3]=\lambda [a_1,a_2,a_3]\).

1.3 Definition "Quadric" (1)
A quadric \(Q\) in \(PG(n-1,q)\) is a primal of order two, so \(Q\) is a quadric, then \(Q = V(F)\), where \(F\) is a quadric form, that is:
\[
F = \sum_{i<j}^{n} a_{ij}x_i x_j = a_{11}x_1^2 + a_{12}x_1 x_2 + \ldots + a_{nn}x_n^2
\]

1.4 Definition "Conics" (1)
Let \(Q(2,q)\) be the set of quadrics in \(PG(2,q)\), that is the varieties \(V(F)\), where:
\[
F = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1 x_2 + a_{13}x_1 x_3 + a_{23}x_2 x_3
\]
If \(V(F)\) is non-singular, then quadric is conic.

1.5 Definition "(k,n)-arcs"
A \((k,n)\)-arc, \(K\) in \(PG(2,q)\) is a set of \(k\) points such that some line in \(PG(2,q)\) meets \(K\) in \(n\) points but such that no line meets \(K\) in more than \(n\) points, where \(n \geq 2\).
A line \(L\) in \(PG(2,q)\) is an i-secant of a \((k,n)\)-arc \(K\) if \(|L \cap K| = i\).
Let \(T_i\) denoted the total number of i-secants to \(K\) in \(PG(2,q)\).
0-secant is called an external line, a 1-secant is called a unisecant, a 2-secant is called a bisecant.

1.6 Definition "Complete (k,n)-arcs" (1)
A \((k,n)\)-arc in \(PG(2,q)\) is complete if there is no \((k+1,n)\)-arc containing it.

1.7 Definition (1)
A point \(N\) not in \((k,n)\)-arc \(K\) is said to be has index \(i\) if there exists exactly \(i\) (2-secants) through \(N\).
\(C_i = |N_i|\) = the number of points of index \(i\).

1.8 Definition "Maximal (k,n)-arcs" (2)
A \((k,n)\)-arc \(K\) in \(PG(2,q)\) is a maximal arc if \(k = (n-1) q + n\).

1.9 Theorem (2)
Let \(M\) be a point of \((k,2)\)-arc \(A\) in \(PG(2,q)\), then the number of unisecant through \(M\) is \(u = q + 2 - k\).

Proof:
There exists exactly \(q + 1\) lines through a point \(M\) in \(a(k,2)\)-arc \(A\) of \(PG(2,q)\), which are the bisecants and the unisecants of the arc. There exists exactly \((k - 1)\) bisecants of the arc \(A\) through \(M\) and the other \((k - 1)\) points of the arc, since the arc contains exactly \(k\) points. The number of unisecants through \(M\) is \(u\), then
\[
u = q + 1 - (k - 1) = q + 1 - k + 1 = q + 2 - k.
\]

1.10 Theorem (2)
Let \(T_i\) be the number of the i-secants of a \((k,n)\)-arc \(A\) in \(PG(2,q)\), then:
(a) \(T_2 = k (k - 1) / 2\)
(b) \(T_1 = k u, u\) is the number of unisecants of each point of \(A\).
(c) \(T_0 = q(q - 1) / 2 + u(u - 1) / 2\).

Proof (a):
\(T_2\) is the number of bisecants of the \((k,n)\)-arc \(A\), the \((k,n)\)-arc \(A\) contains \(k\) points, each two of them determine a bisecant line, so:
\[ T_2 = \binom{k}{2} = \frac{k!}{(k-2)! \cdot 2!} = \frac{k(k-1)}{2} \]

**Proof (b):**

Let \( T_1 \) be the number of unisecants to the \((k,n)\)-arc \( A \). By Theorem (1.6) there exists exactly \( u = q + 2 - k \) lines through any point \( M \) in \((k,n)\)-arc \( A \), since the number of points on \((k,n)\)-arc is \( k \).

Then there exists \( ku = k(q + 2 - k) \) unisecants of the \((k,n)\)-arc \( A \).

**Proof (c):**

Let \( T_0 \) be the number of the external lines to the \((k,n)\)-arc \( A \), then:

- \( T_0 + T_1 + T_2 = q^2 + q + 1 \) represents all the lines in \( PG(2,q) \) then,
- \( T_0 = q^2 + q + 1 - T_1 - T_2 \) from part (a) and (b)
- \( T_0 = q^2 + q + 1 - ku - k(k - 1) / 2 \)

Since, \( u = q + 2 - k \Rightarrow k = q + 2 - u \), then

- \( T_0 = q^2 + q + 1 - u(q + 2 - u) - (q + 2 - u)(q + 1 - u) / 2 \)
- \( T_0 = \frac{1}{2} \left[ 2q^2 + 2q + 2 - 2u(q + 2 - u) - (q + 2 - u)(q + 1 - u) \right] \)
- \( T_0 = \frac{1}{2} \left[ 2q^2 + 2q + 2 - 2u - u^2 - q^2 - q + uq - 2q - 2 + 2u + uq + u - u^2 \right] \)
- \( T_0 = \frac{1}{2} \left[ q^2 - q + u^2 - u \right] \)
- \( T_0 = q (q - 1) / 2 + u(u - 1) / 2 \)

**1.11 Theorem (3)**

A \((k,n)\)-arc \( A \) in \( PG(2,q) \) is complete if and only if \( C_0 = 0 \).

**Proof:** 

Let \( A \) be a complete \((k,n)\)-arc in \( PG(2,q) \) and suppose that \( C_0 \neq 0 \), then \( \exists \) at least one point say \( N \) has an index zero and \( N \not\in A \). Then \( A \cup \{N\} \) is an arc in \( PG(2,q) \). Hence \( A \subseteq A \cup \{N\} \).

Which implies that the \((k,n)\)-arc \( A \) is incomplete (contradicts the hypothesis).

\( \Leftarrow \) suppose that \( C_0 = 0 \) for the \((k,n)\)-arc \( A \) then there are no points of index zero, for \( A \), so the \((k,n)\)-arc \( A \) is a complete.

**1.12 Theorem (3)**

If \( A \) is a maximal arc in \( PG(2,q) \), then,

(a) if \( n = q + 1 \), then \( A = PG(2,q) \)

(b) if \( n = q \), then \( A = PG(2,q) \setminus L \), where \( L \) is line

(c) if \( 2 \leq n \leq q \), then \( n \mid q \) and the dual of the complements of \((k,n)\)-arc \( A \) forms a \((q(q + 1 - n) / n,q / n)\)-arc, also maximal.

**Proof (a):**

A \((k,n)\)-arc \( A \) is a maximal in \( PG(2,q) \), then \( k = (n - 1)q + n \), and if \( n = q + 1 \), then \( k = ((q + 1) - 1)q + (q + 1) = q^2 + q + 1 \) points

\[ A = (q^2 + q + 1,q + 1) = PG(2,q) \]

**Proof (b):**

When \( n = q \), since \( A \) is a maximal arc, then \( A = (n + 1)q + n \), and \( n = (q - 1)q + q = q^2 \)

\[ |PG(2,q)| = q^2 + q + 1 \]

\[ |PG(2,q) \setminus L| = |PG(2,q)| - |L| = q^2 + q + 1 - (q + 1) = q^2 = A. \]
Proof (c):
When \(2 \leq n \leq q\), there exists a point \(M\) not in \(A\), so the number of 0-secants through \(M\) is \(q / n\), it follows that \(n / q\), the dual of complement of \((k,n)\)-arc \(A\) is \((T_0,q / n)\)-arc is maximal. Then \((q(q+1-n) / n,q / n)\)-arc is maximal.

1.13 Lemma (4)
For a \((k,n)\)-arc in \(PG(2,q)\), the following equation hold:

1. \[\sum_{i=0}^{n} T_i = q^2 + q + 1\]
2. \[\sum_{i=1}^{n} iT_i = k (q +1)\]
3. \[\sum_{i=2}^{n} i(i-1)T_i / 2 = k(k - 1) / 2\]
4. \[\sum_{i=2}^{n} (i - 1)p_i = k - 1\]

Note: \(T_i\) denote the total number of \(i\)-secants to the arc in \(PG(2,q)\).

1.14 Theorem (5)
A \((k,n)\)-arc \(A\) in \(PG(2,q)\) is maximal if and only if every line in \(PG(2,q)\) is a 0-secant or \(n\)-secant.

Proof: \(\Rightarrow\) Suppose that \((k,n)\)-arc \(A\) is maximal arc in \(PG(2,q)\), then the result was proved in the theorem.
\(\Leftarrow\) Suppose every line in \(PG(2,q)\) is a 0-secant or \(n\)-secant.
\[\sum_{i=1}^{n} iT_i = k (q +1)\] (by Lemma (1.13), (2))
\[T_1 +2 T_2 + \ldots + (n - 1) T_{n-1} + n T_n = k (q +1)\]
\[n T_n = k (q +1)\] ...[1]
\[\sum_{i=2}^{n} i(i-1)T_i / 2 = k(k - 1) / 2\] (Lemma (1.13), (3))
\[T_2 + 3T_3 + \ldots + n(n - 1) T_{n} / 2 = k(k - 1) / 2\]
\[n(n - 1) T_n / 2 = k(k - 1) / 2\]
\[n(n - 1) T_n = k(k - 1)\] ...[2]
From equation [1], we get:
\[n T_n / k = q + 1\] ...[3]
From equation [2], we get:
\[n T_n / k = (k - 1) / (n - 1)\] ...[4]
From equations [3] and [4], we get
\[(k - 1) / (n - 1) = q + 1 \Rightarrow (k - 1) = (q + 1) (n - 1) \Rightarrow (k - 1) = (n - 1) q + (n - 1)\]
\[\Rightarrow k = (n - 1)q + n\]
\((k,n)\)-arc \(A\) is maximal arc (by definition 1.5)

Section Two
The projective plane \(PG(2,11)\) contains 133 points, 133 lines, every line contains 12 points and every points is on 12 points. The points and lines of \(PG(2,11)\) are shown in table (1,1).
2.1 The Construction of (k,2)-arc in PG(2,11) (2)

Let A = \{(1,2,13,25)\} be the set of unit and reference points in PG(2,11) as in the table (1,1) such that:

\[ I = (1,0,0), 2 = (0,1,0), 13 = (0,0,1), 25 = (1,1,1), A \text{ is (4,2)-arc, since no three points of A are} \]
\[ \text{collinear, the points of A are the vertices of a quadrangle whose sides are the lines.} \]
\[ L_1 = [1,2] = \{1,2,3,4,5,6,7,8,9,10,11,12\} \]
\[ L_2 = [1,13] = \{1,13,14,15,16,17,18,19,20,21,22,23\} \]
\[ L_3 = [1,25] = \{1,24,25,26,27,28,29,30,31,32,33,34\} \]
\[ L_4 = [2,13]= \{2,13,24,35,46,57,68,79,90,101,112,123\} \]
\[ L_5 = [2,25]= \{2,14,25,36,47,58,69,80,91,102,113,124\} \]
\[ L_6 = [13,25] = \{3,13,25,37,49,61,73,85,97,109,121,133\} \]

The diagonal points of A are the points \{3,14,24\} where,
\[ L_1 \cap L_6 = 3; L_2 \cap L_5 = 14; L_3 \cap L_4 = 24. \]

Which are the intersection of pairs of the opposite sides, then there are 61 points on the sides of the quadrangle, four of them are points of the arc A and three of them are the diagonal points of A, so there are 72 points not on the sides of quadrangle which are the points of index zero for A, these points are: 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 62, 63, 64, 65, 66, 67, 70, 71, 72, 74, 75, 76, 77, 78, 81, 82, 83, 84, 86, 87, 88, 89, 92, 93, 94, 95, 96, 98, 99, 100, 103, 104, 105, 106, 107, 108, 110, 111, 114, 115, 116, 117, 118, 119, 120, 122, 125, 126, 127, 128, 129, 130, 131, 132. Hence A is incomplete (4,2)-arc.

2.2 The Conics in PG(2,11) Through the Reference and Unit Points (1)

The general equation of the conic is:
\[ F = a_x x^2 + a_x y^2 + a_y x y + a_2 x^2 + a_3 x y + a_3 y^3 + a_3 x y^2 = 0 \quad \ldots[1] \]
By substituting the points of the arc A in \([1]\), then:
\[ 1 = (1,0,0) \text{ implies that } a_1 = 0, 2 = (0,1,0), \text{ then } a_2 = 0, 13 = (0,0,1), \text{ then } a_3 = 0, \]
\[ 25 = (1,1,1), \text{ then } a_4 + a_5 + a_6 = 0. \]
Hence, from equation \([1]\)
\[ a_4 x_1 x_2 + a_5 x_1 x_3 + a_6 x_2 x_3 = 0 \quad \ldots[2] \]
If \(a_4 = 0\), then \(a_5 x_1 + a_6 x_2 x_3 = 0\), and hence \(x_1(a_5 x_1 + a_6 x_2) = 0\), then \(x_1 = 0\) or \(a_5 x_1 + a_6 x_2 = 0\), which is a pair of lines, then the conic is degenerated, therefore for \(a_4 \neq 0\), similarly \(a_5 \neq 0\) and \(a_6 \neq 0\).

Dividing equation \([2]\) by \(a_4\), one can get:
\[ x_1 x_2 + a_5 x_1 x_3 + a_6 x_2 x_3 = 0, \text{ then } x_1 x_2 + \alpha x_1 x_3 + \beta x_2 x_3 = 0 \quad \ldots[3] \]
where \(\alpha = \frac{a_5}{a_4}, \beta = \frac{a_6}{a_4}\), so that \(1 + \alpha + \beta = 0 \text{ (mod.11)}\)
\[ \beta = -(1 + \alpha), \text{ then } [3] \text{ can be written as} x_1 x_2 + \alpha x_1 x_3 - (1 + \alpha) x_2 x_3 = 0 \]
where \(\alpha 
eq 0\) and \(\alpha 
eq 10\) for if \(\alpha = 0\) or \(\alpha = 10\), then degenerated conics, can be obtained thus \(\alpha = 1, 2, 3, 4, 5, 6, 7, 8, 9\).

2.3 The Equation and the Points of the Conics of PG(2,11) Through the Reference and Unit Points (1)

1. If \(\alpha = 1\), then the equation of the conic \(C_1\) is \(x_1 x_2 + x_1 x_3 + 9 x_2 x_3 = 0\), the points of \(C_1\) are: \{1, 2, 13, 25, 40, 53, 63, 77, 87, 100, 104, 116\}, which is a complete (12,2)-arc, since there are no points of index zero for \(C_1\).

2. If \(\alpha = 2\), then the equation of the conic \(C_2\) is \(x_1 x_2 + 2x_1 x_3 + 8 x_2 x_3 = 0\), the points of \(C_2\) are: \{1, 2, 13, 25, 42, 50, 59, 78, 84, 96, 110, 131\}, which is a complete (12,2)-arc, since there are no points of index zero for \(C_2\).
3. If $\alpha = 3$, then the equation of the conic $C_3$ is $x_1 x_2 + 3x_1 x_3 + 7x_2 x_3 = 0$, the points of $C_3$ are: \{1, 2, 13, 25, 41, 48, 64, 76, 89, 95, 115, 132\}, which is a complete (12,2)-arc, since there are no points of index zero for $C_3$.

4. If $\alpha = 4$, then the equation of the conic $C_4$ is $x_1 x_2 + 4x_1 x_3 + 6x_2 x_3 = 0$, the points of $C_4$ are: \{1, 2, 13, 25, 44, 56, 65, 72, 82, 108, 118, 125\}, which is a complete (12,2)-arc, since there are no points of index zero for $C_4$.

5. If $\alpha = 5$, then the equation of the conic $C_5$ is $x_1 x_2 + 5x_1 x_3 + 5x_2 x_3 = 0$, the points of $C_5$ are: \{1, 2, 13, 25, 43, 51, 67, 71, 99, 103, 119, 127\}, which is a complete (12,2)-arc, since there are no point of index zero for $C_5$.

6. If $\alpha = 6$, then the equation of the conic $C_6$ is $x_1 x_2 + 6x_1 x_3 + 4x_2 x_3 = 0$, the points of $C_6$ are: \{1, 2, 13, 25, 45, 62, 88, 98, 105, 114, 126\}, which is a complete (12,2)-arc, since there are no points of index zero for $C_6$.

7. If $\alpha = 7$, then the equation of the conic $C_7$ is $x_1 x_2 + 7x_1 x_3 + 3x_2 x_3 = 0$, the points of $C_7$ are: \{1, 2, 13, 25, 38, 55, 75, 81, 94, 106, 122, 129\}, which is a complete (12,2)-arc, since there are no points of index zero for $C_7$.

8. If $\alpha = 8$, then the equation of the conic $C_8$ is $x_1 x_2 + 8x_1 x_3 + 2x_2 x_3 = 0$, the points of $C_8$ are: \{1, 2, 13, 25, 39, 60, 74, 86, 92, 111, 120, 128\}, which is a complete (12,2)-arc, since there are no points of index zero for $C_8$.

9. If $\alpha = 9$, then the equation of the conic $C_9$ is $x_1 x_2 + 9x_1 x_3 + 1x_2 x_3 = 0$, the points of $C_9$ are: \{1, 2, 13, 25, 54, 66, 70, 83, 93, 107, 117, 130\}, which is a complete (12,2)-arc, since there are no points of index zero for $C_9$.

Thus there are nine complete (12,2)-arcs (conics) in PG(2,11) through the reference and the unit points. Hence each arc is a maximum arc, since contains (12) points.

2.4 The Construction of Complete (k,n)-arcs in PG(2,11) (2)

1. The construction of complete arcs of degree 3

In 2.3, we found nine complete (k,2)-arcs which are $C_1$, $C_2$, $C_3$, ..., $C_9$, so the complete arcs of degree 3 can be constructed from some complete arcs of degree 2, say $C_1$, $C_1 = \{1, 2, 13, 25, 40, 53, 63, 77, 87, 100, 104, 116\}$. $C_1$ is not complete (k,3)-arc, since there exist some points of index zero for $C_1$ which are \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 54, 55, 56, 57, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 83, 84, 85, 86, 88, 89, 90, 91, 92, 93, 94, 95, 96, 98, 99, 101, 102, 103, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133\}, one can add to $C_1$ seven points of index zero which are: \{12, 14, 45, 49, 57, 70, 128\}, then it can be obtained a complete (19,3)-arc, $H_1 = \{1, 2, 12, 13, 14, 25, 40, 45, 49, 53, 57, 63, 70, 77, 87, 100, 104, 116, 128\}$ since each point not in $H_1$ is on at least one 3-secant and $H_1$ intersect each line in at most 3 points, thus $C_0 = 0$, since there are no points of index zero for $H_1$. Similarly one can find complete arcs of degree 3 from $C_2$, $C_3$, ..., $C_9$, by adding some points of index zero to each one of them, call them: $H_2$, $H_3$, ..., $H_9$.

2. The construction of complete arcs of degree 4

One will try to construct complete arcs of degree 4 from the complete arcs of degree 3, taken the complete (19,3)-arc: $H_1 = \{1, 2, 12, 13, 14, 25, 40, 45, 49, 53, 57, 63, 70, 77, 87, 100, 104, 116, 128\}$, since there exist some points of index zero for $H_1$ which are \{3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 46, 47, 48, 50, 51, 52, 54, 55, 56, 57, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 83, 84, 85, 86, 88, 89, 90, 91, 92, 93, 94, 95, 96, 98, 99\}.
complete arcs of degree 4 from some points of index zero to $H_2$, $H_3$, ..., $H_9$, to obtain complete arcs of degree 4, call them $S_1, S_2, ..., S_9$.

3. The construction of complete arcs of degree 5

In the same method in 1 and 2, one can construct complete arcs of degree 5 by adding some points of index zero to complete arcs of degree 4, for example by taking $S_1$, and the points of index zero for $S_1$: \{3, 4, 5, 6, 7, 8, 9, 11, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37, 39, 41, 42, 44, 46, 48, 50, 51, 52, 54, 55, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 83, 84, 85, 87, 90, 92, 93, 94, 95, 96, 97, 98, 99, 101, 102, 103, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 133\}, and by adding to $S_1$ nine of these points which are: \{8, 22, 27, 43, 56, 62, 74, 85, 112\}, so one can get a complete arc of degree 5 call $M_1$, $M_1$ = \{1, 2, 8, 10, 12, 13, 14, 22, 23, 25, 27, 32, 38, 40, 43, 45, 47, 49, 53, 56, 57, 62, 63, 70, 74, 77, 84, 85, 87, 90, 100, 104, 105, 112, 116, 128\}, $M_1$ is complete arc of degree 5, since there are no point of index zero; i.e. $C_0 = 0$, so every points not in $M_1$ is on at least one 5-secant, and $M_1$ intersects each line in at most 5 points, Similarly one can find complete arcs of degree 5 by adding some point of index zero to $S_2, S_3, ..., S_9$ to obtain complete arcs of degree 5, call them $M_1, M_3, ..., M_9$.

4. The construction of complete arcs of degree 6

Complete arcs of degree 6 can be obtained from the complete arcs of degree 5 by adding some points of index zero, for example, one takes the (36,5)-arc, The points of index zero for $M_1$ are: \{3, 4, 5, 6, 7, 9, 11, 15, 16, 17, 18, 19, 20, 21, 24, 26, 28, 29, 30, 31, 33, 34, 35, 36, 37, 39, 41, 42, 44, 46, 48, 50, 51, 52, 54, 55, 58, 59, 60, 61, 64, 65, 66, 67, 68, 69, 71, 72, 73, 75, 76, 78, 79, 80, 81, 82, 83, 86, 88, 89, 91, 92, 93, 94, 95, 96, 97, 98, 99, 101, 102, 103, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 133\}, and $M_1$ = \{1, 2, 8, 10, 12, 13, 14, 22, 23, 25, 27, 32, 38, 40, 43, 45, 47, 49, 53, 56, 57, 62, 63, 70, 74, 77, 84, 85, 87, 90, 100, 104, 105, 112, 116, 128\}, by adding to $M_1$ eleven of these points which are \{6, 30, 54, 67, 69, 75, 79, 92, 93, 107, 120\}, so we have $N_1$ = \{1, 2, 6, 8, 10, 12, 13, 14, 22, 23, 25, 27, 30, 32, 38, 40, 43, 45, 47, 49, 53, 54, 56, 57, 62, 63, 67, 69, 70, 74, 75, 77, 79, 84, 85, 87, 90, 92, 93, 100, 104, 105, 107, 112, 116, 120, 128\}, then $N_1$ is complete (47,6)-arc, since There are no points of index zero for $N_1$. Similarly one can construct complete arcs of degree 6 by adding some points of index zero to $M_2$, $M_3$, ..., $M_9$, then complete of degree 6 can be obtained, and call them $N_2, N_3, ..., N_9$.

5. The construction of complete arcs of degree 7

Complete arcs of degree 7 can be constructed from the complete arcs of degree 6, one can take the (47,6)-arc, $N_1$ is complete arc of degree 7, since there exist some points of index zero which are: \{3, 4, 5, 7, 9, 11, 15, 16, 17, 18, 19, 20, 21, 24, 26, 28, 29, 31, 33, 34, 35, 36, 37, 39, 41, 42, 44, 46, 48, 50, 51, 52, 55, 58, 59, 60, 61, 64, 65, 66, 68, 71, 72, 73, 76, 78, 80, 81, 82, 83, 86, 88, 89, 91, 94, 95, 96, 97, 98, 99, 101, 102, 103, 106, 108, 109, 110, 111, 113, 114, 115, 117, 118, 119, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 133\}. By adding to $N_1$ eleven of these points which are: \{5, 21, 51, 58, 61, 64, 82, 83, 111, 117, 121\}, then $K_1$ = \{1, 2, 5, 6, 8, 10, 12, 13, 14, 21, 22, 23, 25, 27, 30, 32, 38, 40, 43, 45, 47, 49, 51, 53, 54, 56, 57, 58, 61, 62, 63, 64, 67, 69, 70, 74, 75, 77, 79, 82, 83, 84, 85, 87, 90, 92, 93, 100, 101, 103, 104, 105, 112, 115, 116, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 133\}.
is a complete (58,7)-arc, since there are no points of index zero, thus every point not in \( K_1 \) is on at least one 7-secant and \( K_1 \) intersects each line in at most 7 points. Similarly, constructed arcs of degree 7 can be constructed from \( N_2, N_3, \ldots, N_9 \), call them \( K_2, K_3, \ldots, K_9 \).

6. The construction of complete arcs of degree 8

Complete arcs of degree 8 can be constructed from the complete arcs of degree 7, one can take the (58,7)-arc, \( k_1 \) is complete (58,7)-arc, since there exist some points of index zero which are: \{3, 4, 5, 7, 9, 11, 15, 16, 17, 18, 19, 20, 24, 26, 28, 29, 31, 33, 34, 35, 36, 37, 39, 41, 42, 44, 46, 48, 50, 52, 55, 59, 60, 61, 64, 65, 66, 68, 71, 72, 73, 76, 78, 80, 81, 86, 88, 89, 91, 92, 94, 95, 96, 97, 98, 99, 101, 102, 103, 106, 108, 109, 110, 113, 114, 115, 118, 119, 122, 123, 124, 125, 126, 127, 130, 131, 132, 133\}. By adding to \( k_1 \) thirteen of these points which are: \{3, 16, 24, 26, 28, 35, 37, 41, 48, 59, 78, 98, 125\}, to obtain a complete (71,8)-arc \( L_1 \) and \( L_1 = \{1, 2, 3, 5, 6, 8, 10, 12, 13, 14, 16, 21, 22, 23, 24, 25, 26, 27, 28, 30, 32, 35, 37, 38, 40, 41, 43, 45, 46, 47, 49, 51, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 67, 69, 70, 74, 75, 77, 79, 82, 83, 84, 85, 87, 90, 92, 93, 98, 100, 104, 105, 107, 111, 112, 116, 117, 120, 121, 125, 128\} is a complete (71,8)-arc, since there are no points of index zero, thus every point on \( L_1 \) is on at least one 8-secant and \( L_1 \) intersects any line in at most 8 points. Similarly arcs of degree 8 can be constructed from \( K_2, K_3, \ldots, K_9 \), call them \( L_2, L_3, \ldots, L_9 \).

7. The construction of complete arcs of degree 9

Complete arcs of degree 9 can be constructed from the complete arcs of degree 8, the complete (71,8)-arc \( L_1 \) is taken, \( L_1 \) is in complete (71,9)-arc, the points of index zero of \( L_1 \) are: \{4, 7, 9, 11, 15, 17, 18, 19, 20, 29, 34, 36, 39, 42, 44, 46, 50, 52, 55, 60, 65, 66, 68, 71, 72, 73, 76, 80, 81, 86, 88, 89, 91, 94, 95, 96, 97, 99, 101, 102, 103, 106, 108, 109, 110, 113, 114, 115, 118, 119, 122, 123, 124, 126, 127, 129, 130, 131, 132, 133\}. By adding to \( L_1 \) twelve of these points which are: \{4, 15, 29, 36, 44, 52, 65, 71, 80, 88, 119, 133\}, then a complete (83,9)-arc call it \( O_1 \) is obtained \( O_1 = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 35, 37, 38, 40, 41, 43, 44, 45, 47, 48, 49, 51, 52, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 65, 67, 69, 70, 74, 75, 77, 79, 80, 82, 83, 84, 85, 87, 88, 90, 92, 93, 98, 100, 104, 105, 107, 111, 112, 116, 117, 120, 121, 125, 128\} is a complete (83,9)-arc, since there are no points of index zero, thus every point on \( O_1 \) is on at least one 9-secant and \( O_1 \) intersects any line in at most 9 points. In the same way complete arcs of degree 9 can be obtained from \( O_2, O_3, \ldots, O_9 \).

8. The construction of complete arcs of degree 10

Complete arcs of degree 10 can be constructed from the complete arcs of degree 9 as the following:
The complete arc of degree 9, \( O_1 \) is complete (83,10)-arc, since there exist some points of index zero for \( O_1 \) which are: \{7, 9, 11, 17, 18, 19, 20, 31, 33, 34, 39, 42, 44, 46, 50, 55, 60, 66, 68, 72, 73, 76, 81, 86, 89, 91, 94, 95, 96, 97, 99, 101, 102, 103, 106, 108, 109, 110, 113, 114, 115, 118, 122, 123, 124, 126, 127, 129, 130, 131, 132\}. Twelve of these points are added to \( O_1 \) which are: \{9, 17, 31, 42, 46, 73, 86, 95, 96, 99, 103, 113\}, then a complete (95,10)-arc call it \( B_1 \), is obtained \( B_1 = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 17, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 51, 52, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 65, 67, 69, 70, 71, 73, 74, 75, 77, 78, 79, 80, 82, 83, 84, 85, 86, 87, 88, 90, 92, 93, 95, 96, 98, 99, 100, 103, 104, 105, 107, 111, 112, 113, 116, 117, 119, 120, 121, 125, 128, 133\} is a complete (95,10)-arc, since there are no points of index zero, i.e. \( C_0 = 0 \).
Similarly complete arcs of degree 10 can be constructed, call it $B_1, B_2, \ldots, B_9$ from $O_2, O_3, \ldots, O_9$.

9. Them construction of complete arcs of degree 11

Complete arcs of degree 11 can be constructed from complete arcs of degree 10. The complete arcs of degree 10 $B_i$ is taken. $B_i$ is in complete $(95,11)$-arc, since there exist some points of index zero for $B_i$ which are: \(7, 17, 11, 13, 23, 27, 29, 30, 31, 32, 33, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 47, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 77, 78, 79, 80, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 98, 99, 100, 101, 103, 104, 105, 106, 107, 108, 109, 110, 112, 114, 115, 116, 117, 118, 120, 121, 122, 124, 125, 126, 127, 128, 129, 130, 131, 132\), by adding to $B_i$ (26) points of these points which are: \(\{11, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 47, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 77, 78, 79, 80, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 98, 99, 100, 101, 103, 104, 105, 106, 107, 108, 109, 110, 112, 114, 115, 116, 117, 119, 120, 122, 124, 125, 126, 127, 128, 129, 130, 131, 132\}\].

Reference

5. بان عبد الكريم (2001)، رسالة ماجستير، جامعة الموصل، العراق.

\[\text{Table : } (1,1) \text{ of the points and lines of } \text{PG}(2,11)\]

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إنشاء الأقواس الكاملة $(k,n)$ في المستوى الإسقاطي $(11,2)$ حول حقل كالو $(11)$ حيث إن $11 \leq n \leq 3$

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الخلاصة

إن الغاية الأساسية من هذا البحث هو إيجاد توس كامل $(k,n)$ في الفضاء الإسقاطي الثنائي $(k,n)$ حسب حقل كالو $(11)$ وفقاً لوصفتة إضافة بعض النقاط ذاتياً صغرى إلى التفاصيل الكامل $(1,k,n-1)$ حيث $11 \leq n \leq 3$.

القوس $(k,n)$ هو مجموعة $k$ من النقاط ليس هناك $n + 1$ على استقامة واحدة.

القوس الكامل $(k,n)$ هو توس لا يمكن أن يكون محتوى في التفاصيل $(k+1,n)$. 