An Inverse Scattering Problem
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Abstract:
In this paper we give an elementary method to study an inverse scattering problem for a pair of Hamiltonians (H(h), H₀(h)) on L²(IRⁿ), where H₀(h) = −h²Δ, H(h) = H₀(h) + V, and V is a short-range potential. We show that, in dimension n ≥ 3, the scattering operators S(h) : h ∈ (0, h₀] which are localized near a fixed energy ̂λ > 0 determine the asymptotic of the potential V at infinity. This approach can be used to solve an inverse scattering problem for isotropic external metrics.

1. Introduction:
In this paper we study an inverse scattering problem for a pair (H(h), H₀(h)) on L²(IRⁿ), n ≥ 2 where the free operator
H₀(h) = −h²Δ; h ∈ (0, h₀] is the semi-classical parameter with h₀ small enough and
H(h) = H₀(h) + V, (1)
where V ∈ C∞₀(IRⁿ) and satisfies ∀ α ∈ INⁿ,
\[ \tilde{\partial}_α^2 V(x) ≤ C_α < x >^{−n+|α|}, \rho > 1. \] (a)
under the hypothesis (a) the wave operators:
W⁺(H(h), H₀(h)) = s − lim \[ \int_{−∞}^{+∞} e^{it\tilde{H}(h)} e^{−it\tilde{H}_0(h)} \] (2)
exist and complete, i.e. Ran
W⁺(H(h), H₀(h)) = H₀(H) = subspace of absolute continuity of H(h) [8]. Let S(h) be the scattering operator defined by S(h) = W⁺(H(h), H₀(h))W⁻(H(h), H₀(h)) (3)
In order to localize the scattering operator near a fixed energy ̂λ > 0, we introduce a cut-off function\nb \[ x ∈ C₀∞(0, + ∞), x = 1 \] in a neighborhood of ̂λ > 0.
The goal of this note is to obtain some information on the potential from S(h)(x(H₀(h)), h) in the semi-classical limit h → 0. We show that for n ≥ 3, the operators S(h)x(H₀(h)), h ∈ (0, h₀] determine the asymptotic of the potential at infinity, by studying the asymptotic of:
F(h) = ⟨S(h)x(H₀(h))Φₘₙ, Ψₘₙ⟩ > (4)
where ⟨, ⟩ is the usual scalar product in L²(IRⁿ), and Φₘₙ, Ψₘₙ are suitable test functions.

2. Semi-classical asymptotics for the localized scattering operator and application:
2.1 Definition of the test functions:
The dilation operator U(h²), δ > 0, on L²(IRⁿ) can be defined as follows [2]:
\[ U(h²)Φ(x) = h² Φ(hδx), \] (5)
we also need an energy cut-off x₀ ∈ C₀∞(IRⁿ) such that
\[ x₀(ξ) = 1 \] if \[ |ξ| ≤ 1, x₀(ξ) = 0 \] if \[ |ξ| ≥ 2. \]
For w ∈ Sⁿ−¹, we write x ∈ IRⁿ as
\[ x = y + tw, y ∈ Π_w = \text{orthogonal hyperplane to } w \] and we consider:
\[ X_w = \{ x = y + tw \in IR^n : |y| ≥ 1 \}. \] (6)
Now, we can define for \( Φ ∈ C₀∞(X_w) \) and suitable δ, ε > 0,
\[ Φ_{h,w} = e^{−\frac{1}{δ} \sqrt{t^w} w} U(h^δ) x₀(h^δ D) Φ, \] (7)
where D = −i∇, (Ψₘₙ) is defined in the same way with \[ Ψ ∈ C₀∞(X_w)). \]

2.2 Semi-classical asymptotics for the scattering operator:
In this section we prove the following theorem:
Theorem 2.2.1:
\[ \langle (S(h) - 1)x(H₀(h))Φₘₙ, Ψₘₙ⟩ > = \frac{1}{2i\sqrt{δ}} \int_{−∞}^{∞} V(x₀ + tw) dΦΨ > + o(h^n). \] (8)
where \[ μ = δ(ρ − 1) > 1 > 0. \]
Remark:
Using (a), it is easy to see that the first term of the (R.H.S) is equal to \( O(h^n) \) since Φ, Ψ have their support in \( X_w \).

Proof:
Step 1:
Let begin by an elementary lemma [8]
Lemma 2.2.1:
\[ ∀ ε < 1 + δ, ∀ h ∈ (0, h₀], \text{ we have } : \]
\[ x(H₀(h))Φₘₙ, Ψₘₙ = Φ_{h,w} \] (9)
we easily obtain:
\[ F(x(H₀(h))Φₘₙ, Ψₘₙ) = h^{−\frac{d}{2}} x₀(h^{−\frac{d}{2}} (h^δξ − √δw)) F(Φ, Ψ) > + o(h^n). \] (10)
where F is the usual Fourier transform then, on Supp \( x₀ \), we have
\[ |hξ − √δw| ≤ 2h^{1+δ−ε}. \]
so, for \( ε < 1 + δ \) and \( h₀ \) small enough, we have \( x₀((hξ)^{2}) = 1. \)
Then by Lemma 2.2.1, we obtain
\[ F(h) = ⟨W⁺(H(h), H₀(h))Φₘₙ, W⁺(H(h), H₀(h))Ψₘₙ⟩ > \]
and calculation gives
\[ F(h) = ⟨Ω(h, w)x₀, (h^δ D)Φ, Ω⁺(h, w)x₀(h^δ D)Ψ >, \] (12)
where
\[ \Omega^z(h, w) = s - \lim_{t \to \pm \infty} e^{i H(t, h, w)} e^{-i H(h, w)} \]

(13)

with

\[ H_0(h, w) = (D + \sqrt{A}) \ h^{-((i + \delta) w)} \]  \hspace{1cm} \text{(14)}

and

\[ H(h, w) = H_0(h, w) + h^{-2(i + \delta)} V(h^{-\delta} x). \]  \hspace{1cm} \text{(15)}

so by (12) we have to find the asymptotic of \( \Omega^z(h, w) x_\delta(h^z D) \Phi \). We follow the same strategy as in [6], [7] and only treat the case (+).

**Step 2:**

we construct a modifier \( J^+(h, w) \) in the form as in [6], [7]

\[ J^+(h, w) = 1 + h^\delta d^-(h^{-\delta} x, w), \]  \hspace{1cm} \text{(16)}

where \( \nu \) a suitable parameter defined below, we denote:

\[ J^+(h, w) = H(h, w) J^-(h, w) - J^-(h, w) H(h, w). \]  \hspace{1cm} \text{(17)}

A direct calculation shows that

\[ J^+(h, w) = h^{\nu} e^{-V(h^{-\delta} x)} - 2\sqrt{A} h^{1/2} \nu \nabla (d^-(h^{-\nu} w)), \]  \hspace{1cm} \text{(18)}

thus, we choose \( \nu = -1 \) and we solve the transport equation

\[ w \nabla d^- (x, w) = \frac{1}{2i \sqrt{A}} V(x). \]  \hspace{1cm} \text{(19)}

The solution of (19) is given by

\[ d^- (x, w) = \frac{i}{2 \sqrt{A} h} \int (x + tw\mu) d\nu dt. \]  \hspace{1cm} \text{(20)}

we obtain

\[ \int_{\nu} d^{-\nu} (h^{-\nu} x, w) \leq C_{\nu} h^{\nu-\nu-1} \]  \hspace{1cm} \text{(21)}

where

\[ \Gamma^- = \{ (x, \xi) : x \geq R, \xi \geq a, x \xi \geq -\sigma \mid x \mid \xi \}, \sigma \in (-1,1). \]

From this, we deduce the following lemma [6].

**Lemma 2.2.3:**

\[ \Omega^z(h, w) x_\delta(h^z D) \Phi = \lim_{t \to +\infty} e^{i \text{arg}(h, w)} J^+(h, w) e^{-i \text{arg}(h, w)} x_\delta(h^z D) \Phi. \]

(22)

**Step 3:**

Now, we can formulate [8]

**Lemma 2.2.3:**

For \( \delta > \frac{1}{\rho - 1} \) and \( \varepsilon < 1 + \delta, \rho \neq 1 \), we have

\[ \left\| \Omega^z(h, w) - J^+(h, w) x_\delta(h^z D) \Phi \right\| = o(h^{\delta - \rho - 1}). \]

(23)

First, we write

\[ \Omega^z(h, w) - J^+(h, w) x_\delta(h^z D) \Phi = \int_{\nu} e^{i \text{arg}(h, w)} J^+(h, w) e^{-i \text{arg}(h, w)} x_\delta(h^z D) \Phi d\nu. \]

(24)

In order to introduce a cut-off, which localizes far from the origin, let \( V_\nu \) be a neighborhood of \( w \) in \( S^\omega \). We define \( O^+ = \{ x + tw: x \in \text{Supp} \Phi, t \geq 0, w \in V_\nu \} \).

(25)

If \( V_\nu \) is rather small, it is clear that \( O^+ \subset IR^n \setminus B \) where \( B \) is the unit ball. Let \( X^+ \subset C^\infty(IR^n \setminus B) \) be a cut-off function such that \( x^+ \equiv 1 \) in a conical neighborhood of \( O^+ \). Now, we have the following estimation which is obtained by using a standard non stationary phase argument

\[ \forall \varepsilon < 1 + \delta, \forall N \geq 1, (x^+ - 1) e^{-i \text{arg}(h, w)} x_\delta(h^z D) \Phi = O(\varepsilon < 1 + \delta, h^\delta) \]

(26)

In the sense of the \( L^2 \)-norm [7].

So, we deduce

\[ \left\| \Omega^z(h, w) - J^+(h, w) x_\delta(h^z D) \Phi \right\| \leq \int_{\nu} \left\| e^{i \text{arg}(h, w)} J^+(h, w) e^{-i \text{arg}(h, w)} x_\delta(h^z D) \Phi \right\| d\nu, \]  \hspace{1cm} \text{(27)}

Modulo \( O(h^\rho) \). Since \( Supp x \subset IR^n \setminus B \), we obtain using (21) and (27) as in [7]

\[ \left\| \Omega^z(h, w) - J^+(h, w) x_\delta(h^z D) \Phi \right\| = O(h^{\delta - \rho - 1} + O(h^{\rho - 1})), \]  \hspace{1cm} \text{(28)}

and the last term is equal to \( O(h^{\delta - \rho - 1}) \) if \( \delta > \frac{1}{\rho - 1} \), : \( \rho \neq 1 \).

**Step 4:**

Using the following estimation on \( \mathcal{L}^3(\mathbb{R}^n): \forall N \geq 1, \) \( (x_\delta(h^z D) - 1) \Phi = O(h^N), n \geq 1 \) \hspace{1cm} \text{(29)}

and Lemma 2.2.3, we obtain

\[ \Omega^z(h, w) x_\delta(h^z D) \Phi = (1 + \frac{i}{2\sqrt{A}} d^- (h^{-\delta} x, w) + O(h^{\delta - \rho - 1})), \]  \hspace{1cm} \text{(30)}

**2.3 An application to an inverse scattering problem in semi-classical asymptotics:**

We use equation (8) in the particular case where \( V \) is an asymptotic homogeneous function. Let \( V_j, j = 1, 2 \) be two potentials satisfying when \( \mid x \mid \to +\infty \)

\[ V_j(x) = \mid x \mid^{-\rho} f_j(x) \mid x \mid^\rho, \rho > 1, \]  \hspace{1cm} \text{(31)}

where \( f_j \in C^\infty(S^{n-1}), S^{n-1} \) being the unit sphere of \( IR^n \), we denote \( S_j(h) \) the scattering operator associated with the pair \( (H(h)+V_j, H(h)) \). We have the following result

**Corollary 2.3.1:**

For \( n \geq 3, \) assume that \( \forall h \in (0, \kappa_h], S_j(h)x(H_0(h)) = S_j(h)x(H_0(h)) \), then \( f_j = f_2 \).

**Proof:**

We have for \( \delta > \frac{1}{\rho - 1} \) and \( \varepsilon < 1 + \delta, \rho \neq 1 \), by Theorem 2.2.1

\[ (S_j(h) - 1)x(H_0(h)) \Phi = x(H_0(h)) \Phi, \forall \varepsilon, \nu \geq \frac{h^\delta}{2\sqrt{A}} \int_\nu \int_{\nu} V_\nu(x + tw) d\nu d\nu \leq o(h^\rho). \]

(32)

where \( V_\nu = \int_\nu f_j(x) \int_\nu (x + tw) d\nu \). So, if

\[ S_j(h)x(H_0(h)) = S_2(h)x(H_0(h)), \forall h \in (0, \kappa_h], \]

we deduce

\[ \forall w \in S^{n-1}, \forall x \in X_w \int_{\nu} \int_{\nu} V_\nu(x + tw) d\nu = 0, \]

(33)

where \( V_\nu = V_{\nu_1} - V_{\nu_2}. \) So, using the support theorem for the Radon transform [3], [6] we obtain \( V_\nu(x) = 0, \mid x \mid \geq 1. \)

**Remark:**

In a non semi-classical context (\( h = 1 \)), let us mention the reference [5] where the inverse scattering problem at a fixed energy is treated. They showed that if \( V_1, V_2 \in S_{t=1}^2(IR^n), n \geq 3 \) and if the associated matrices at some non-zero fixed energy are equal up to smooth terms then \( V_1 - V_2 \in S^{-\infty}(IR^n) \).

**3. An inverse scattering problem in the case of isotropic external metrics:**

**3.1 Notations:**
We show that the previous approach can be used to solve an inverse scattering problem for perturbation of order 2 of the free Laplacian $H_0 = -\Delta$.

Let us consider the following Hamiltonian on $L^2(\mathbb{R}^n), n \geq 2$:

$$H = \sum_{i,j} D_i g^i(x) D_j,$$  \hspace{1cm} (34)

where $G(x) = (g^i(x))$ is a $C^\infty$-definite positive metric satisfying

$$\forall \alpha \in \mathbb{N}, \begin{bmatrix} \partial_x \end{bmatrix}^\alpha (G(x) - I) \leq C_{\alpha} x^{-\alpha - 1}, \rho > 1. \hspace{1cm} (b)$$

so, we can define the wave operators $\{1\}$

$$W^\pm = s - \lim_{t \to \pm \infty} e^{itH} e^{-itH_0}, \hspace{1cm} (35)$$

and the scattering operator $S = W^+ W^-$.

In order to study the asymptotic at high energies of the scattering operator, we consider the following test function

$$\Phi_{\lambda,w} = e^{i \sqrt{\lambda} x \cdot \chi}\left( \lambda - \frac{\delta}{2} \right) x_0 \left( \lambda - \frac{\delta}{2} \right) D \Phi,$$  \hspace{1cm} (36)

where

$$\Phi \in C_c^\infty(X_+), (\Psi_{\lambda,w} \text{ is defined in the same way with } \Psi \in C_c^\infty(X_+)) \cdot$$

In [2] Enss and Weder also used such test functions with $\delta = 0, \varepsilon > 0$. The uniqueness of inverse potential scattering problems is given in [6] and [7]. We have the following result where $(\lambda, \omega)$ is the scalar product in $I^{n+1}$ and $H(x) = G(x) - I$:

**Theorem 3.1.1:**

For $\delta > \frac{1}{\rho - 1}$ and $\varepsilon < 1 + \delta, \rho \neq 1$ we have, when $\lambda \to +\infty$,

$$\langle (S - 1)\Phi_{\lambda,w}, \Psi_{\lambda,w} \rangle = \frac{\sqrt{\lambda}}{2\pi} \int_{\mathbb{R}^n} \langle (H(\lambda^2 x + N)w, w) e\Phi, \Psi \rangle > 0$$

(37) \hspace{1cm} where

$$\mu = \delta(\rho - 1) - 1 > 0.$$ \hspace{1cm} \text{Proof:}

As in section 2, we define

$$F(\lambda) = \langle W^+ \Phi_{\lambda,w}, W^- \Psi_{\lambda,w} \rangle.$$ \hspace{1cm} \text{We see that:}

$$F(\lambda) = \langle \Omega^+(h, w)x_0(h^{-1}D)e\Phi, \Omega^-(h, w)x_0(h^{-1}D) \Psi \rangle,$$  \hspace{1cm} (38)

where $\Omega^\pm (h, w)$ is defined by (29) with $h = \lambda^\frac{1}{2}$ and

$$H(h, w) = \sum_{i,j} (D_i + h^{-1-i/2}) w_j g^i(h^{-1} x)(D_j + h^{-1-i/2}) w_j.$$  \hspace{1cm} (39)

so, everything done in section 2 also works in this situation.

### 3.2 An application to an inverse scattering problem of isotropic external metrics:

We consider isotropic homogeneous metrics $G_{ij} = 1, 2$ satisfying (b) and when $|x| \to +\infty$

$$G_j(x) = \begin{bmatrix} x \end{bmatrix}^\rho f_j(\begin{bmatrix} x \end{bmatrix}^\rho) I d |x|^\rho, \rho > 1$$  \hspace{1cm} (40)$$

where $f_j \in C^\infty(S^{n-1})$. Let $S_j$ be the associated scattering operator. As in section 3, we have:

**Corollary 3.2.1:**

In dimension $n \geq 3$ we have

$$S_1 = S_2 \Rightarrow f_1 = f_2 \hspace{1cm} (41)$$

**Proof:**

We have for $\delta > \frac{1}{\rho - 1}$ and $\varepsilon < 1 + \delta$, by Theorem 1

$$\langle (S - 1)\Phi_{\lambda,w}, \Psi_{\lambda,w} \rangle = \frac{\sqrt{\lambda}}{2\pi} \int_{\mathbb{R}^n} \langle (H(\lambda^2 x + N)w, w) e\Phi, \Psi \rangle > 0,$$

(42) \hspace{1cm} where

$$G_{ij} = \begin{bmatrix} x \end{bmatrix}^\rho f_j(\begin{bmatrix} x \end{bmatrix}^\rho).$$

So, if

$$S_j(h) = \begin{bmatrix} x \end{bmatrix}^\rho f_j(\begin{bmatrix} x \end{bmatrix}^\rho) \phi_{1, \rho}, \Psi_{1, \rho} \geq 0,$$

we deduce

$$\forall w \in S^{n-1}, \forall \chi \in X_+, \int_{\mathbb{R}^n} G_0(x + N) dt = 0,$$  \hspace{1cm} (43).$$

So, using the support theorem for the Radon transform [3], [6] we obtain

$$G_0(x) = 0, |x| \geq \frac{1}{\varepsilon}.$$ \hspace{1cm} □

### 4. Discussion:

The present work deals with the inverse scattering problem of a pair of Hamiltonians $(H(h), H_0(h))$ on $L^2(\mathbb{R}^n)$. Section 2 discussed the approximated semi classical technique for the localized scattering operators through finding the wave operator, in addition to formulate the scattering operator equation in dimension $n \geq 3$ which centered near bounded energy $\lambda > 0$.

From the scattering operator equation we determine the approximated potential at infinity, in addition to some applications and this is clear from corollary 2.3.1. Section 3 discussed the inverse scattering operators for isotropic external metrics, in addition to finding the wave operators which lead us to finding the scattering operator in dimension $n \geq 3$, also we mention some applications on this case which is clear from corollary 3.2.1.
References:

معضلة الاستطارة العكسية
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الملخص:
في هذا البحث تم إعطاء الطريقة الأساسية لدراسة مسألة الاستطارة العكسية لزوج من المؤثرات الهاملتونية (H(h), H₀(h)) على (IRⁿ, L²(IRⁿ)) في الابد h ∈ (0, h₀), H₀(h) = -h²∆، H(h) = H₀(h) + V، H₀(h) = - h²∆ وتمثل V الجهد القصير المدى ، فقد تم إثبات أن مؤثرات الاستطارة (S(h)) في الابد 3 ≥ n = 3، H ∈ (0, h₀) تتميز قريب طاقة محدودة في قيمتها λ والتي تحدد الجهد التجريبي عند الابتدائية، إن هذه الطريقة يمكن استخدامها لحل مسألة الاستطارة العكسية لأي معيار متساوي الخواص .