ON $m$–Topological space
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Abstract:
In this paper, we study the $m$–Compact on $m$–Topological spaces, and we introduce a same new $m$–separation axioms of $m$–Topological spaces $(m-T_0, m-T_1, m-T_2)$ and we proof all $m$– separation axioms are $m$– hereditary and $m$–Topological property.

1- Introduction:
Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces on which $m$–separation axioms $(T_0, T_1, T_2)$ are assumed unless explicitly stated [3]. A sub class $\tau^* \subseteq \tau$ is called supratopology on $X$ if $X \in \tau^*$ and $\tau^*$ is closed under arbitrary union, let $(X, \tau^*)$ is called a supratopological space.
The members of $\tau^*$ are called supra open sets. We called $\tau^*$ asupratopology associated with $\tau_X$ if $\tau_X \subseteq \tau^*$ let $(X, \tau^*_X)$ and $(Y, \tau^*_Y)$ be supratopological. A function $f : (X, \tau^*_X) \rightarrow (Y, \tau^*_Y)$ is an $S^*$–continuous function if the inverse image of each supra open set in $Y$ is a supra open set in $X$ [1]. let $E$ be a subset of $X$, $E$ is called an $m$–set with $\tau^*$ if $E \cap T \in \tau^*$ for all $T \in \tau^*$. Then the class $\tau^*_m$ of all $m$–sets with $\tau^*$ is contained $\tau^*$ called an $m$– topology with $\tau^*$ and the members of $\tau^*_m$ are called $m$– open sets. A subset $B$ of $X$ is called an $m$– closed set if the complement of $B$ is an $m$–open set. Thus the intersection of any family of $m$–closed sets is a $m$–closed set. in case $\tau^*_m$ is an $m$– topology with $\tau^*$ on $X$ the topological spaces $(X, \tau^*_X, \tau^*_Y)$ with $\tau^*$ be denoted by $(X, \tau^*_m)[5]$. The $m$–closure (resp. $m$–interior) of a subset $E$ of $X$ will be denoted by $m-\text{CL}(E)$ (resp. $m-\text{int}(E)$) is the intersection of all $m$– closed subset of $X$ containing $E$ (resp. the union of all $m$– open subsets of $X$ which is contained in $E$). We say that a function $f : (X, \tau^*_X) \rightarrow (Y, \mu^*_Y)$ is called $m$–open function. If the image of any $m$–open set in $X$ is an $m$–open set in $Y$, we say that $f$ is a $S^*$– homeomorphism if and only if $f$ is bijective, $f$ is supra open function and $f$ is $S^*$–continuous [5] let $P$ be any property in $X$, if $P$ is carried by $S^*$–home to another space $Y$ we say $P$ is a topological property. Let $A$ be subset of $X$, $A$ $m$–cover of $A$ is a family of subsets of $X$ whose union includes $A$. A $m$– sub cover of $A$ $m$–cover of $A$ is a sub family of so $A$ $m$– cover of $A$.

Lemma 1.1.
Let $f : (X, \tau^*_m) \rightarrow (Y, \mu^*_m)$ be is $S^*$–continuous function, then function is $mS$– continuous.

2- $m-T_*$ space induced by $m$–Topology.
Definition 2.1.
Let $(X, \tau^*_m)$ be an $m$– topological space, then $(X, \tau^*_m)$ is called $m$– $T_*$ space and denoted by $(m-T_*)$ if for any distinct pair of points $x, y$ of $X$ there exists one $m$–open set $U$ in $\tau^*_m$ contains one of the points but not the other.

Example 2.2.
Let $X = \{a, b, c, e\}$ and
$$\tau^* = \{X, \{a\}, \{c\}, \{e\}, \{a, c\}, \{a, e\}, \{a, c, e\}, \{a, b, c\}\}$$
with $\phi$ then
$$\tau^*_m = \{X, \phi, \{a\}, \{c\}, \{e\}, \{a, c\}, \{a, e\}, \{a, c, e\}, \{a, b, c\}\}$$
is $m-\tau^*_0$.
And we take $\tau^*$ is supratopology without empty set thus
$$\tau^*_m = \{X, \{a, b, c\}\}$$is not $m-\tau^*_0$.

Remark 2.3
Every $m$–open set on $(X, \tau^*_m)$ is asupraopenset on $(X, \tau^*)$ the converse is not true.

Example 2.4.
Let $X = \{a, b, c, e\}$,
$$\tau^* = \{X, \phi, \{a\}, \{c\}, \{e\}, \{a, c\}, \{a, e\}, \{a, c, e\}, \{a, b, c\}\}$$
and
$$\tau^*_m = \{X, \phi, \{a\}, \{c\}, \{e\}, \{a, c\}, \{a, e\}, \{a, c, e\}, \{a, b, c\}\}$$
notice that each of $\{a, b, c\}, \{a, b, c\}$ is supopen set but not $m$–open set.

Theorem 2.5.
An $m$–Topological space $(X, \tau^*_m)$ is $m-\tau^*_0$– space if and only if for each pair of distinct points $x, y$ of $X$, $m-\text{cl}^*(\{x\}) \neq m-\text{cl}^*(\{y\})$.

Proof:
Sufficiency. Suppose that $x, y \in X$, $x \neq y$. Let $z \in X$ such that $z \in m-\text{cl}^*(\{x\})$ but $z \not\in m-\text{cl}^*(\{y\})$. We claim that $x \not\in m-\text{cl}^*(\{y\})$, for $x \in m-\text{cl}^*(\{y\})$ then $m-\text{cl}^*(\{x\}) \subset m-\text{cl}^*(\{y\})$, this contradiction the fact that $z \not\in m-\text{cl}^*(\{y\})$, consequently $x \in (m-\text{cl}(y))^c$ to which $y$ does not belong.
Necessity let \((X, \tau_m)\) be an \(m-T_m\)-space and \(x , y \in X , x \neq y, \exists m - open\) set \(u_m \ni x \in u_m or y \in u_m\) then \(u_m^y\) is an \(m\)-closed set which \(x \in u_m\) and \(y \in u_m^y\). Since \(m-c l (\{y\})\) is the smallest \(m\)-closed set containing \(y\) [because \(m-c l (E) = E \cup (m- \text{int}(E) )\], if \(m-c l (\{y\}) \subset u_m\) and therefore \(x \notin m - c l (\{ y\})\). The \(m - c l (\{x\}) \neq m-c l (\{y\})\).

**Definition 2.6:**

Let \((X, \tau_m)\) be an \(m\)-topological space, let \(E\) be a subset of \(X\) then the \(\tau_m = \{ (E \cap T_m) \in \tau_m \backslash T_m \} \) is \(m\)-open set, is called relative \(m\)-topology space \((m - subspace for short)\).

**Example 2.1.**

Let \(X = \{a,b,c,e\}, \tau^*\) is supratopology of with empty set and also \(E = \{a,b,c\}\) then \(\tau_{m_e} = \{E, \phi, \{a\}, \{a,c\}, \{b,c\}, \{b,a\}\}\), hence \((E, \tau_{m_e})\) is called relative \(m\)-sub space.

**Definition 2.7:**

Let \((X, \tau_m)\) be any \(m\)-topological space, if \(p\) is any property in \(X\) then we called \(p\) is \(m\)-hereditary if its appear in a relative \(m\)-topological space if no we say \(p\) is non-\(m\)-hereditary.

**Theorem 2.8:**

Let \((X, \tau_m)\) be any \(m-T_m\) - space, then the relative \(m\)-topology space \((E, \tau_{m_e})\) is \(m-T_m\).

**Proof:**

Since \((X, \tau_m)\) is the \(m\)-topology space of \(m - T_0\), let \(e_1 \neq e_2 \in \exists X\) \(m\)-openset \(u_m \ni X\) such that \(u_m\) contains one of \(e_1, e_2\) but not both, since \(E \subseteq X\) let \(x_1, x_2 \in E\), \(e_1 \neq e_2\) now we have \(e_1 \in E \). \(e_2 \in u_m\) then \(e_1 \in E \cap u_m = u_m \) or \(e_2 \in E\) and \(e_2 \in E \) then \(e_2 \in E \cap u_m = u_m \) hence is \(m - T_m\) - space.

**Definition 2.9:**

A function \(f : (X, \tau_m) \to (Y, \mu_y)\) is \(m\)-homeomorphism if and only if \(f\) is bijective, \(m\)-open function and \(m\)-continuous.

**Definition 2.10.**

Let \(f : (X, \tau_m) \to (Y, \mu_y)\) be an \(m\)-homeomorphism, let \(p\) any property in \(X\) we say that \(p\) is \(m\)-topological property if \(p\) is appear in \(Y\). **Theorem 2.11.**

The property \(m - T_m\) on \(m\)-topology space is topological property.

**Proof:**

Let \((X, \tau_m), (Y, \mu_y)\) be an \(m\)-topological spaces \(f : (X, \tau_m) \to (Y, \mu_y)\) a function be \(m\)-homeomorphism.

Let \(y_1 \neq y_2 \in Y\) since \(f\) is bijective, \(\exists y_1 \neq y_2 \in X\) such that \(y_1 = f(x_1), y_2 = f(x_2)\) since \((X, \tau_m)\) is \(m-T_m\)-space, then \(\exists\) one \(m\)-openset \(u_m\) of \(X\) such that \(x_1 \in u_m\), \(x_2 \notin u_m\) or \(x_1 \notin u_m\), \(x_2 \in u_m\) and function \(m\)-open, then \(f(x_1) \in f(u_m)\), \(\forall x_1 \in f(u_m)\), \(x_1 \in u_m\) hence \((Y, \mu_y)\) is \(m-T_m\)-space.

**3 - \(m-T_1\) - space induced by \(m\)-topology.**

**Definition 3.1:**

Let \((X, \tau_m)\) be an \(m\)-topological space, then \((X, \tau_m)\) is called \(m-T_1\)-space and denoted by \((m-T_1)\) if for any distinct pair of points \(x, y\) of \(X\) there exists two \(m\)-open sets \(u_m\), \(v_m\) in \(\tau_m\) such that, \(x \in u_m\), \(y \notin v_m\) and \(y \notin u_m\), then \((m-T_1)\) space.

**Remark 3.2:**

Every \(m-T_1\)-spaces is \(m-T_m\)-spaces but the converse is not true according. From example (2.2) \((X, \tau_m)\) is \(m-T_m\) - spaces but not \(m-T_1\).

**Theorem 3.3:**

An \(m\)-Topological space \((X, \tau_m)\) is \(m-T_1\)-space if and only if every singleton subset of \(X\) is \(m\)-closed.

**Proof:**

Suppose \(X = m-T_1\)-space and \(x \in X\) we show that \(\{x\}^c\) is \(m\)-open, let \(y \in \{x\}^c\) then \(x \neq y\), so by \(m-T_1\) there exist an \(m\)-openset \(G_{y}, s.t. x \in G_{y}\) but \(y \notin G_{y}\) hence \(x \in G_{y} \subseteq \{x\}^c\) and \(\{x\}^c = \bigcup \{G_{y} : x \in \{x\}^c\}\).

Conversely, suppose \(\{x\}^c\) is \(m\)-closed for every \(x \in X\) let \(x \neq y \in X\) and \(x \neq y\) implies \(x \in \{y\}^c\) is an \(m\)-open set and \(y \notin \{y\}^c\) is an \(m\)-open set. To show that \((X, \tau_m)\) is \(m-T_1\)-space, since \(\{x\}^c, \{y\}^c\) are \(m\)-open sets, \(x \in \{y\}^c\) and \(y \notin \{x\}^c\) then \(m-T_1\)-space.

**Proposition 3.4.**

Let \((X, \tau_m)\) be any \(m-T_1\)-space, then the relative \(m\)-topological space \((E, \tau_{m_e})\) is \(m-T_1\).

**Proof:**

since \((X, \tau_m)\) be an \(m\)-topology space of \(m-T_1\)-space, let \(e_1, e_2 \in X\), \(\exists\) two \(m\)-open sets \(u_m\),
Theorem 3.6.
The property \( m - T_1 \)-space is topological property.

Proof:
Let \( (X, \tau_m), (Y, \mu_m) \) be \( m \)-topology spaces 
\[
f : (X, \tau_m) \to (Y, \mu_m)
\]
be function is \( m \)-home. Let \( y_1 \neq y_2 \in Y \) since \( f \) is abjective, \( \exists x_1, x_2 \in X \) such that 
\[
y_1 = f(x_1), y_2 = f(x_2)
\]
since \( (X, \tau_m) \) is \( m - T_1 \). \( \exists \) two an \( m \)-open sets \( u_m, v_m \) of \( X \) such that 
\[
x_1 \in u_m, x_2 \notin u_m \quad \text{and} \quad x_2 \in v_m, x_1 \notin v_m.
\]
And \( m \)-open function then 
\[
f(x_1) = y_1 = f(u_m)
\]
and \( f(x_2) = y_2 = f(v_m) \) is \( m \)-open hence \( (Y, \mu_m) \) is \( m - T_1 \).

4- \( m - T_2 \)-space induced by \( m \)-topology.

Definition 4.1.
Let \( (X, \tau_m) \) be an \( m \)-topological space, then \( (X, \tau_m) \)
is called \( m - T_2 \)-space and denoted by \( m - T_2 \) if for any distinct pair of points \( x, y \) of \( X \) there exists two disjoint \( m \)-open sets \( u_m, v_m \) in \( \tau_m \) contains them respectively. For example

Let \( X = \{a, b, c, e\} \), \( \tau^* \) is supratopology of \( X \) with empty set
\[
\tau_m = \{X, \phi, \{a\}, \{b\}, \{a, c\}, \{b, a\}, \{e, a\}, \{b, a, e\}, \{b, c\}, \{b, e\}, \{c, a\}\}
\]
is \( m - \tau_1 \) but not \( m - \tau_2 \)

Theorem 4.4.
Let \( (X, \tau_m) \) be any \( m - T_2 \) space, then the relative \( m \)-topology space \( (E, \tau_{m_E}) \) is \( m - T_2 \)-space.

Proof:
Since \( (X, \tau_m) \) be an \( m \)-topology space of \( m - T_2 \)-space, let \( e_1 \neq e_2 \in X \), \( \exists \) two disjoint \( m \)-open sets \( u_m, v_m \) of \( X \), such that \( e_1 \in u_m, e_2 \notin u_m \) and \( e_2 \in v_m, e_1 \notin v_m \). Let \( E \subseteq X \), \( e_1 \neq e_2 \in E \) now we have \( e_1 \in E \), \( e_1 \in u_m \) then \( e_1 \in E \cap u_m = u_{m_E} \) and \( e_2 \in E \), \( e_2 \in v_m \) then \( e_2 \in E \cap v_m = v_{m_E} \). To prove \( u_{m_E} \cap v_{m_E} = \phi \) since 
\[
u_{m_E} \cap v_{m_E} = (E \cap u_m) \cap (E \cap v_m) = E \cap (u_m \cap v_m) = E \cap \phi = \phi
\]
then \( (E, \tau_{m_E}) \) is \( m - T_2 \)-space.

Theorem 4.5.
The property \( m - T_2 \)-space is topological property.

Proof:
Let \( (X, \tau_m), (Y, \mu_m) \) be an \( m \)-topology spaces, since \( f : (X, \tau_m) \to (Y, \mu_m) \), a function is \( m \)-home. Let \( y_1 \neq y_2 \in Y \) since \( f \) is abjective, \( \exists x_1, x_2 \in X \), \( x_1 \neq x_2 \) such that \( y_1 = f(x_1), y_2 = f(x_2) \). Since \( (X, \tau_m) \) is \( m - T_2 \), \( \exists \) two disjoint \( m \)-open sets \( u_m, v_m \) of \( X \) continuing the respectively. Sines \( m \)-open function then
\[
f(x_1) = y_1 \in f(u_m) = u_{m_*}
\]
and \( f(x_2) = y_2 \in f(v_m) = v_{m_*} \) are \( m \)-open in \( Y \), since's \( f^{-1} \) is \( m \)-continuous hence \( u_{m_*} \cap v_{m_*} = f(u_m) \cap f(v_m) = f(u_m \cap v_m) = f(\phi) = \phi \) then \( (Y, \mu_m) \) is \( m - T_2 \).

A \( m - T_2 \) is \( m \)-compact if each \( m \)-open covering has affine \( m \)-sub covering.

Example 4.7.
Let \( X = \{a, b, c, e\} \), \( \tau^* \) is supratopology of \( X \) with empty set,
\[
\tau_m = \{[X, \phi, \{a\}, \{b\}, \{a, c\}, \{b, a\}, \{e, a\}, \{b, a, e\}, \{b, c\}, \{b, e\}, \{c, a\}\}
\]
is clearly every \( m - T_2 \)-space is \( m \)-compact. Hence
\[
m - T_2 \text{ is } m \text{- compact if and only if is finite.}
\]

Theorem 4.8.
\( m \)-compactness is a topological property.

Proof:
Let \( (X, \tau_m) \) be an \( m \)-compact space. Since \( f : (X, \tau_m) \to (Y, \mu_m) \) be function is \( m \)-home. To show that \( (Y, \mu_m) \) is \( m \)-compact space. let \( \{u_m \} \) be an \( m \)-open cover \( Y \). Then \( Y \subseteq \bigcup u_m \) since's \( f \) is \( m \)-continuous, \( f^{-1}(u_m) = v_m \) hence \( v_m \) are \( m \)-open sub set of \( X \). Since \( X \) is \( m \)-compact and 
\[
x \subseteq \bigcup v_m \Rightarrow y = f(x) \in \bigcup u_m \Rightarrow y \in u_m \text{ then } (Y, \mu_m) \text{ is } m \text{- compact space.}
\]

Theorem 4.9.
Let \( (X, \tau_m) \) is \( m \)-compact space if and only if for each family \( \{H_\alpha : \alpha \in I\} \) of \( m \)-closed sets in \( X \)
satisfying $\bigcap_{a=1}^{n} H_a = \phi$, there is a finite sub family $H_{a_1}, ..., H_{a_m}$ with $\bigcap_{i=1}^{m} H_{a_i} = \phi$.

**Proof:**
Suppose $(X, \tau_m)$ is $m$-compact space, let $\forall \{H_\alpha : \alpha \in I\}$ of $m$-closed sets in $X$, $\bigcap_{\alpha} H_\alpha = \phi$. Then by De Morgan’s law $X = \bigcup_{\alpha} H_\alpha^c$, so $\{H_\alpha^c\}$ is $m$-open cover of $X$, since each $H_\alpha^c$ is $m$-closed. But $X$ is $m$-compact, hence $\exists H_{a_1}^c, H_{a_2}^c, ..., H_{a_m}^c \subseteq \{H_\alpha^c\}$ s.t. $X = H_{a_1}^c \cup H_{a_2}^c \cup ... \cup H_{a_m}^c$ thus by De Morgan’s law, $\phi = \bigcap_{i=1}^{n} H_{a_i}$.

**Conversely.** Let $\{G_\alpha\}$ be an $m$-open cover of $X$, $X = \bigcup_{\alpha} G_\alpha$ by De Morgan’s law $x = x = (\bigcup_{\alpha} G_\alpha)^c = \bigcap_{\alpha} G_\alpha^c$. Since each $G_\alpha$ is $m$-open, $\{G_\alpha^c\}$ is a class of $m$-closed set. Hence $\exists G_{a_1}^c, G_{a_2}^c, ..., G_{a_m}^c$ s.t. $X = G_{a_1}^c \cup G_{a_2}^c \cup ... \cup G_{a_m}^c$ thus by De Morgan’s law, $\phi = \bigcup_{i=1}^{m} G_{a_i}$.

**Proposition 4.10.**
Any $m$-closed subspace of $m$-compact space is $m$-compact.

**Proposition 4.11.**
Every $m$-compact subset of $m-T_2$-space is $m$-closed.

**Proof:**
Let $K$ be an $m$-compact subset of $m-T_2$-space of $X$. Let $x \in X \setminus K$. For each $y \in K$, $\exists$ disjoint $m$-open $U_y$ and $V_y$ of $Y$ and $X$ respectively. then $\{U_y\}$ is an $m$-open cover $K$ which to a finite sub covering $\{U_{y_i}\}_{i=1}^{n}$, say $K$ is $m$-compact. Let $V_i$ be the $m$-open of $X$ for $i = 1, 2, ..., n$ then $\bigcap_{i=1}^{n} V_i$ is $m$-open of $X$ and $\bigcap_{i=1}^{n} U_{y_i} = \phi, \forall i = 1, 2, ..., n$ implies that $x \not\in V \subseteq X \setminus K$ this $X \setminus K$ is $m$-open and $K$ is $m$-closed.

**Proposition 4.12.**
The image of any $K$ $m$-closed subset of $m$-compact space is $m$-closed is $m-T_2$-space under $ms$-continuous.

**Proof:**
By **Proposition 4.10** $K$ is $m$-compact space if $f : (X, \tau_m) \rightarrow (X, \tau_m)$ is $ms$-continuous, then $f(K)$ is $m$-compact by (4.11) hence $m$-closed, $m-T_2$-space.

**Theorem 4.13.**
Let $(X, \tau_m)$ be an $m$-compact, $Y$ be $m-T_2$-space and $f : (X, \tau_m) \rightarrow (Y, \mu_m)$ $ms$-continuous then $f$ is $m$-closed map.

**Proof:**
Let $A \subseteq X$ be an $m$-closed it is $m$-compact and consequently so is $f(A)$ since $Y$ is $m-T_2$-space, then $f(A)$ is $m$-closed in $Y$. 


References

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حول الفضاءات التبولوجيه

منا بهجت ياسين
قسم الرياضيات , كلية التربية للبنات , جامعة تكريت , تكريت , العراق

المتخص

في هذا البحث درسنا تراص على الفضاءات التبولوجية ود ع نا تعريفا جديدا لبعض بديهيات الفصل وبرناها بان كل بديهيات الفصل - والصفة التبولوجية $(m - T_0, m - T_1, m - T_2)$