On Weakly $π$–Regular Rings

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Abstract
The main purpose of this paper is to study right(left)-weakly $π$-regular ring. Also we give some properties of a weakly $π$-regular ring, and the connection between such rings and CS-ring, MP-ring and Quasi-Duo ring, and finally we gives the discrimination of this ring with the generalized right principally injective modules and simple singular right $R$-module which is GP-injective.

1. Introduction:
Throughout this paper, $R$ represents an associative rings with identity and all right(left) $R$-module are unitary. $J(R)$ denotes the Jacobson radical of a ring $R$. A ring $R$ is said to be right(left) weakly regular if $a \in aRa$ for every $a \in R$, $R$ is weakly regular ring if it is both right and left weakly regular [1]. A ring $R$ is said to be $π$-regular if for every element $a$ in $R$ there exists a positive integer $n=n(a)$ depending on $a$, such that $a^n=aa^n$. A ring $R$ is said to be reduced if $R$ has no non-zero nilpotent element [5]. A ring $R$ is said to be right(left) generalized semi regular ring if for all $a \in R$, and there exists $b \in R$, and there exists a positive integers $n$ such that $a^n=aa^n$, and $r(a^n)=r(b)$ $(\text{Id}_a^n) = (\text{Id}_b)$. For every $a \in R, r(a)$ and $1(a)$ will stand respectively for right and left annihilators of $a$. A right $R$-Module is called generalized right principally injective (briefly right GP- injective) if for any $0 \neq a \in R$, there exists a positive integer $n$ such that $a^n \neq 0$, and any right $R$-homomorphism of $a^R$ in to $M$ extends to one of $R$ in to $M$.[6].

2. Weakly $π$–Regular Rings (Basic properties):
A ring $R$ is said to be right(left) weakly $π$-regular ring if for every element $a$ in $R$, there exists a positive integer $n=n(a)$ depending on $a$, such that:
$$a^n \in a^R \text{Ra} (a^n \in Ra^R).$$
A ring $R$ is said to be weakly $π$-regular ring if it is both right and left weakly $π$-regular.[4].

Clearly every $π$-regular ring is weakly $π$-regular ring, but the converse is not true, if $R$ is a commutative weakly $π$-regular ring, then $R$ is $π$–regular ring, also if $R$ is reduced then every right weakly $π$-regular ring is left weakly $π$-regular ring.

Lemma 2.1: [3]
The $J(R)$ Jacobson radical of a right weakly $π$-regular rings are nil.

Lemma 2.2: Let $R$ be a right weakly $π$–regular ring, then $R=Ra^R$ (for all $a^R$ is not a zero divisor ) and $n$ is a positive integer depending on $a$.

Proof: Let $a^R$ is not a zero divisor of the ring $R$, since $R$ is a right weakly $π$–regular ring, then $a^nR=(aR)^n=a^R \text{Ra}$, hence $a^n(R-Ra^R)=0$ and $(R-Ra^R) \subseteq r(a^n)$.
Since $r(a^n)=L(a^n)=0$, ( $a^n$ is not a zero divisor). Therefore $R=Ra^R$.

Proposition 2.3: Let $R$ be a ring with a condition that for all non-zero elements $a$ in $R$,there exists a positive integer $n=n(a)$ depending on $a$, such that $r(a^n) \cap Ra^R=0$. Then:

1- If for every $b,c \in R$, $(1- ba^c) \in Ra^R$, then every right weakly $π$–regular ring is a left weakly $π$–regular ring.
2- Every right weakly $π$–regular ring is a right generalized semi regular ring.

Proof(1):
Let $0 \neq a \in R$ then there exists a positive integer $n(n(a))$ depending on $a$, such that $r(a^n) \cap Ra^R=0$, since $R$ is a right weakly $π$–regular ring, then $a^n \in a^R \text{Ra}$, let $b,c \in R$, then $a^n= a^nb^c$, then $a^n(1- ba^c )=0$, then $1- ba^c \in r(a^n)$.
Since for every $b,c \in R$ $(1- ba^c ) \in Ra^R$, then there exist $x \in Ra^R$ such that:
$$1- ba^c=x \in Ra^R,$$then $1= ba^c+x \in Ra^R$, therefore $R=Ra^R$.
Then $(1- ba^c )\in r(a^n) \cap Ra^R=0$, $(1- ba^c ) = 0$, hence $1= ba^c$, implies $a^n= ba^ca^n$.
Therefore $a^n \in Ra^R$, then $R$ is a left weakly $π$–regular rings.

Proof(2):
Let $0 \neq a \in R$ then there exists a positive integer $n(n(a))$ depending on $a$, such that $r(a^n) \cap Ra^R=0$, since $R$ is a right weakly $π$–regular ring, then $a^n \in a^R \text{Ra}$, let $b,c \in R$, then $a^n= a^nb^c$, let $d = ba^c$, implies $a^n= a^d$, we must show that $r(d) = r(a^n)$, let $x \in r(d)$, then $dx=0$, (multiply by $a^n$ from left) then $(a^n)d=0$.
Since $(a^n= d)$ then $a^n x = 0$, therefore $x \in r(a^n)$, then $r(d) \subseteq r(a^n)$.
Let $x \in r(a^n)$, then $a^n x = 0$, since $d= ba^c \in Ra^R$, and $dx \in Ra^R$, and $a^n= a^d$, then $a^n dx=0$, implies $dx \in r(a^n)$, therefore $dx \in r(a^n) \cap Ra^R=0$, then $dx =0$, $x \in r(d)$, we get that $r(a^n) \subseteq r(d)$, therefore $R$ is a right generalized semi regular ring.

Lemma 2.4: Let $R$ be a right weakly $π$–regular ring, then $\text{Cent}(R)$ is $π$–regular.

Proof: Let $a \in \text{Cent}(R)$, since $R$ is a right weakly $π$–regular ring, then for every element $a$ in $R$, there exists a positive integer $n=n(a)$ depending on $a$, such that:
$$a^n= a^R \text{Ra}$$, let $b,c \in R$, then $a^n= a^nb^c= a^nc^a$, and so we have that $a^n= ba^ca^n$.
Let $d = bc$, implies $a^n= a^nb^n$, therefore $\text{Cent}(R)$ is $π$–regular.

Theorem 2.5: Let $I$ be a right weakly regular ideal and $R/I$ be a right weakly $π$–regular ring, then $R$ is a right weakly $π$–regular ring.
Proof: Let $R/I$ be a right weakly $\pi$-regular ring, then for all $a$ in $R$ there exists a positive integer $n$ such that $a^n + I = I$ and there exists $t, w$ in $I$, such that:

$$a^n + I = (a^n + 1) + (a^n + 1) + (a^n + 1) + \ldots \in I,$$

and there exists $t, w$ in $I$, such that:

$$a^n + a^n r a^n s = (a^n - a^n r a^n s) I,$$

and there exists $t, w$ in $I$, such that:

$$a^n + a^n r a^n s = (a^n - a^n r a^n s) I.$$

Theorem 3.2: Let $R$ be a right weakly $\pi$-regular ring. Then $R$ is a right weakly $\pi$-regular ring.

Proof: Let $R$ be a right weakly $\pi$-regular ring and to prove that $R$ is a right weakly $\pi$-regular ring. Let $Ra^R + r(a)=R$, if not then there exists a maximal right ideal $M$ containing $Ra^R + r(a)$ such that $Ra^R + r(a) \subseteq M$, since $R$ is a right MP ring, then every maximal right ideal of $R$ is a left pure ideal (since every left pure ideal is generalized left pure ideal). Then for all $a \in M$, exists $b \in M$ and a positive integer $n$ such that $a^n = ba^n$, then $a^n - ba^n = 0$, implies $(1-b)a = 0$, then $(1-b) \in L(a) \subseteq R$, hence $1 \in M$, a contradiction.

Therefore, $R$ is a right weakly $\pi$-regular ring.

3. The Connection Between Weakly $\pi$-Regular Rings and Other Rings:

In this section, we study the connection between Weakly $\pi$-regular rings and CS-ring, MP-ring. Quasi-Duo ring, the right R-Module which generalized right principally injective (briefly right GP- injective module), and simple singular right R-module which is GP-injective.

Recall that a ring $R$ is said to be right (left) CS-ring if every non-zero right (left) ideal is essential in a direct summand [2], equivalently, every right (left) closed ideal is a direct summand, clearly every maximal right ideal is right closed. An ideal $I$ of a ring $R$ is said to be right(left) pure ideal if for all $a \in I$, there exists $b \in I$ such that $a = ab(ba)$. An ideal $I$ is said to be right(left) Generalized pure ideal if for all $a \in I$, there exists $b \in I$, and there exists a positive integers $n$ such that $a^n = ba^n(ba)^n$. A ring $R$ is said to be right(left) MP-ring if every maximal right(left) ideal is a left(right) pure ideal. A ring $R$ is called right(left) Quasi-Duo ring if every maximal right(left) ideal of $R$ is two sided ideal[3].

Theorem 3.1: Let $R$ be a right CS-ring with for all element $a$ in $R$, $L(a^n)$ is two sided ideal then $R$ is a left weakly $\pi$-regular ring.

Proof: Let $R$ be a right CS-ring and to prove that $R$ is a right weakly $\pi$-regular ring. Let $Ra^R + L(a^n) = R$, if not then there a maximal right ideal $M$ containing $Ra^R + L(a^n)$, such that $Ra^R + L(a^n) \subseteq M$, since $R$ is a right CS-ring, then every maximal right ideal of $R$ is a direct summand such that:

$$M \subseteq K = R,$$

Hence $(Ra^R + L(a^n)) \subseteq M \cap K = 0$, then $(Ra^R + L(a^n)) \cap K = 0$, also $Ra^R \cap K = 0$, $L(a^n) \cap K = 0$, then $Ka^n \in Ra^R$ and $Ka^n \in K$, therefore $Ka^n \in Ra^R \cap K = 0$, then $Ka^n = 0$, then $K \subseteq L(a^n)$, a contradiction. Thus $Ra^R + L(a^n) = R$.

Let $b, c$ be any two elements in $R$, and $d \in L(a^n)$ and $1 \in R$, then $ba^nc + d = 1$, then $ba^nc = a^n$, therefore $a^n = ba^nc = a^n$, then $a^n \in a^n Ra^R$, then $R$ is a right weakly $\pi$-regular ring.

Corollary 3.3: Let $R$ be a ring with every maximal right ideal is a right generalized pure ideal then $R$ is a right weakly $\pi$-regular ring.

Proof: We must show that $R$ is a right weakly $\pi$-regular ring, let $Ra^R + r(a^n) = R$, if not then there a maximal right ideal $M$ containing $Ra^R + r(a^n)$ such that $Ra^R + r(a^n) \subseteq M$, since $R$ is a right MP ring, then every maximal right ideal of $R$ is a left pure ideal (since every left pure ideal is generalized left pure ideal). Then for all $a \in M$, exists $b \in M$ and a positive integer $n$ such that $a^n = ba^n$, then $a^n - ba^n = 0$, implies $(1-b)a = 0$, then $(1-b) \in L(a^n) \subseteq R$, hence $1 \in M$, a contradiction.

Therefore, $R$ is a right weakly $\pi$-regular ring.

Lemma 3.4: Let $R$ be a right weakly $\pi$-regular ring, then every two sided ideal is a right generalized pure ideal.

Proof: Let $R$ be a right weakly $\pi$-regular ring, then $a^n \in a^n Ra^R$, there exists $b, c \in R$, such that $a^n = ba^nc$, let $I$ be a two sided ideal of $R$. Then for all $a \in I$, there exists a positive integers $n$ such that $a^n \in I$ (is an ideal of $R$), then $a^n \in I$ (is a two sided ideal of $R$).

Let $d = ba^nc \in I$, then $a^n = a^nd$, therefore $I$ is a right generalized pure ideal.

Lemma 3.5: If $R$ is quasi-duo ring and left MP-ring, then $R$ is a right weakly $\pi$-regular ring.

Proof: We must show that $R$ is a right weakly $\pi$-regular ring, that is meaning $Ra^R + r(a^n) = R$, if not then there a maximal right ideal $M$ containing $Ra^R + r(a^n)$ such that $Ra^R + r(a^n) \subseteq M$, since $R$ is left MP-ring (every maximal left ideal is a right pure ideal) and every right pure ideals are right generalized pure ideals, then there exists $a$ and $b$ in $M$ and there exists a positive integer $n$ such that $a^n = a^n b$, then $ba^nc = a^n$, then $a^n \in Ra^R a^n$, then $R$ is a right weakly $\pi$-regular ring.
then $a^n \ (1- b)=0$, then $(1-b) \in (a^n) \subseteq M$, then $1 \in M$ a contradiction. Hence $Ra^n R^+ r(a^n)=R$. Let $b, c$ be any two elements in $R$, and $d \in r(a^n)$ and $1 \in R$, then $ba^c d=1$.

Then $a^n ba^c c + a^n d=a^n$, therefore $a^n ba^c c = a^n$, then $a^n \in a^n Ra^n R$, then $R$ is a right weakly $\pi$-regular ring.

**Lemma 3.6:** [7] 
If $R$ is left(right) quasi-duo ring and $J(R)=0$, then $R$ is reduced ring.

**Corollary 3.7:** If $R$ is quasi-duo ring and $J(R)=0$ and left MP-ring, then $R$ is a weakly $\pi$-regular ring.

**Proof:** 
By using (Lemma 3.5) $R$ is a right weakly $\pi$-regular ring, by using (Lemma 3.6) $R$ is reduced ring, then $R$ is a weakly $\pi$-regular ring.

**Theorem 3.8:** If $R$ is ring with every simple right $R$-module is GP-injective, then $R$ is a right weakly $\pi$-regular ring.

**Proof:** 
Let $Ra^n R^+ r(a^n)=R$, if not then there a maximal right ideal $M$ containing $Ra^n R^+ r(a^n)$ such that: $Ra^n R^+ r(a^n) \subseteq M$.

We define $f: a^n R \to R/M$ by $f(a^n r)=r+M$, for every element $r$ in $R$, $f$ is a well defined function. Indeed if $a^n r_1 \ a^n r_2$ belong to $a^n R$, and $a^n r_1=a^n r_2$, then $a^n r_2-a^n r_1=0$, implies that $a^n (r_1 - r_2)=0$, then $r_1 - r_2 \in r(a^n) \subseteq M$, hence $r_1 - r_2 \in M$, then $r_1 +M=r_2 +M$.

Then $f(a^n r_1)=f(a^n r_2)$.

Since $R/M$ is simple module then $R/M$ is GP-injective module, then exists $f$: $a^n R \to R/M$ such that $f(a^n r)=r+M = (c+M) a^n$, then $r+M=ca^n +M$, let $r=1$, then $1+M=ca^n +M$, from this, we get $1- ca^n \in M$. Since $ca^n \in Ra^n R \subseteq M$, then $1 \in M$.

A contradiction. Hence $Ra^n R^+ r(a^n)=R$, let $b, c$ be any two elements in $R$, and $d \in r(a^n)$ and $1 \in R$, then $ba^c d=1$.

Then $a^n ba^c c + a^n d=a^n$, therefore $a^n ba^c c = a^n$, then $a^n \in a^n Ra^n R$, then $R$ is a right weakly $\pi$-regular ring.

**Lemma 3.9:** Let $R$ be a ring with for all elements $a$ in $R$ there exists a positive integer $n$ such that $L(a^n) \subseteq r(a^n)$, then $Ra^n R^+ r(a^n)$ is essential right ideal of $R$.

**Proof:** 
Let $a$ be any element of a ring $R$, and assume that $Ra^n R^+ r(a^n) \cap 1=0$, where $1$ be a non zero right ideal of $R$, then $Ra^n R^+ r(a^n) \cap 1=0$ , since $La^n \subseteq Ra^n R$, $La^n \subseteq 1$, then $La^n \subseteq Ra^n R^+ r(a^n)$, $La^n \subseteq 1$, then $I \subseteq L(a^n) \subseteq r(a^n)$, $I \subseteq r(a^n)$, then $I=0$, a contradiction. Therefore, $Ra^n R^+ r(a^n)$ is essential right ideal of $R$.

**Theorem 3.10:** If $R$ is a ring with every simple singular right $R$-module is GP-injective, and for all element $a$ in $R$, $L(a^n) \subseteq r(a^n)$, where $n$ is a positive integers then $R$ is a right weakly $\pi$-regular ring.

**Proof:** 
We must show that $R$ is a right weakly $\pi$-regular ring, that is meaning that $Ra^n R^+ r(a^n)=R$, if not then there is a maximal right ideal $M$ containing $Ra^n R^+ r(a^n)$ such that: $Ra^n R^+ r(a^n) \subseteq M$, then by (Lemma 3.9), $Ra^n R^+ r(a^n)$ is essential right ideal of $R$, then $R/M$ is GP-injective module, then there exist a positive integers $n$ such that any $R$-homomorphism of $a^n R$ in $R/M$ extends to one of $R$ in to $R/M$, we define $f$: $a^n R \to R/M$ by $f(a^n r)=r+M$, for every elements $r$ in $R$, $f$ is a well defined function because that if $a^n r_1 , a^n r_2$ belongs to $a^n R$, and $a^n r_2 =a^n r_2$ , then $a^n (r_1 - r_2)=0$, implies $a^n (r_1 - r_2)=0$.

Then $(r_1 - r_2) \in r(a^n) \subseteq M$, hence $(r_1 - r_2) \in M$, then $r_1 +M=r_2 +M$.

Thus $f(a^n r_1)=f(a^n r_2)$.

Let $(r=1)$, then $1+M=ca^n +M$. Thus $1- ca^n \in M$, and hence $1 \in M$ , a contradiction.

Therefore $Ra^n R^+ r(a^n)=R$ , let $b, c$ be any two elements in $R$, and $d \in r(a^n)$ and $1 \in R$, then $ba^c d=1$.

Then $a^n ba^c + a^n d=a^n$, therefore $a^n ba^c = a^n$, then $a^n \in a^n Ra^n R$, then $R$ is a right weakly $\pi$-regular ring.
References
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