Pseudo-Weakly-N-Quasi-Injective Modules

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Abstract
Let R be a commutative ring with unity and M be a unitary R-module. An R-module M is said to be N-injective where N is an R-module if \( f(N) \subseteq M \) for each \( f \in \text{Hom}(N,E(M)) \), where E(M) is the injective hull of M. And M is called weakly-N-injective if for each \( f \in \text{Hom}(N,E(M)) \), there exists a submodule X of E(M) such that \( f(N) \subseteq X \subseteq M \). In this paper we give generalizations for the concepts N-injective and weakly-N-injective modules. We call an R-module M pseudo-N-injective if \( f(N) \subseteq M \) for each monomorphism \( f: N \to \overline{M} \) where \( \overline{M} \) is quasi injective hull of M. And we call M is Pseudo-weakly-N-injective module if for each monomorphism \( f: N \to \overline{M} \), there exists a submodule X of \( \overline{M} \) such that \( f(N) \subseteq X \subseteq M \). Our main goal in this work is to study the basic properties of these concepts, and give examples, characterizations of pseudo-weakly-N- quasi-injective and study the relation of these concepts with other modules.

Introduction
Let R be a commutative ring with unity and M , N be two R-modules. M is said be N-injective if \( f(N) \subseteq M \) for each \( f \in \text{Hom}(N,E(M)) \), where E(M) is the injective hull of M. This concept was introduced first by Azumaya [ 3]. Weakly-N-injective module was introduced originally by [10] as a generalization of N-injective module. Since then the study of this concept has been extensively in[9],[11]. We introduce in this paper generalizations of both N-injective module and weakly-N-injective module. We call an R-module M pseudo-N- quasi-injective modules if \( f(N) \subseteq M \) for each monomorphism \( f: N \to \overline{M} \) where \( \overline{M} \) is quasi injective hull of M .And we call An R-module M is Pseudo- weakly-N- quasi-injective module if for each monomorphism \( f: N \to \overline{M} \), there exists a submodule X of \( \overline{M} \) such that \( f(N) \subseteq X \subseteq M \). Also we introduce a new concept named pseudo-invertible submodule to prove that if M is a torsion free R-module and N is pseudo-invertible submodule of M such that M is pseudo-weakly-M/N- quasi-injective module. Then N is pseudo-invertible submodule of \( \overline{M} \).

1- Pseudo-N-Quasi-injective modules
In this section we introduce the definition of pseudo-N- quasi-injective module as a generalization of N-injective module and gives some basic properties, examples of this concept.

Definition 1.1
Let M and N be two R-modules. M is called pseudo-N- quasi-injective modules if \( f(N) \subseteq M \) for each monomorphism \( f: N \to \overline{M} \), where \( \overline{M} \) is a quasi- injective hull of M

Examples and remarks 1.2
1. Every N-injective module is pseudo-N-quasi-injective module, but the converse is not true , as the following example shows: \( Z^3_3 \) as a Z-module is pseudo-N-quasi-injective module , but not Z-injective, for if \( f: Z \to E(Z_3) = Z_3^\infty \) defined by \( f(n) = \frac{n}{3^2} + Z \) for all \( n \in Z \). It is clear that \( f(Z) = Z_3 \) which is not embeds in \( Z_3^\infty \).
2. \( Z_2 \) as a Z-module is pseudo-Z-quasi-injective module.
3. Every injective module is pseudo-N-quasi-injective modules for any R-module N.
4. Every quasi-injective module is pseudo-N-quasi-injective module for any R-module N.
5. Every pseudo-injective module is pseudo-N-quasi- injective module for any R-module N.
6. Every semi-simple R-module is pseudo-N-quasi- injective module for any R-module N.
7. M is pseudo-injective if and only if M is pseudo-\( \overline{M} \)- quasi-injective module.
8. Z as a Z-module is pseudo-Z-quasi-injective module.
9. If M is pseudo-N-quasi-injective module and L is a submodule of N , it not necessary that M is pseudo-L-quasi-injective module, as the following example shows:
From(8) Z as a Z-module is pseudo-Z-quasi-injective module. but Z is not pseudo-3Z-quasi-injective module, for if \( g: 3Z \to \overline{Z} = \overline{Q} \) is monomorphism define by \( g(3n) = 3n/7 \) for all n in Z, but \( g(3Z) \) is not embeds in Z. Before we give the next proposition, we need to recall the following definitions.
A non-zero submodule K of an R-module M is said to be an essential submodule of M if \( L \cap K \neq 0 \) for every non-zero submodule L of M. And we said that M is an essential extension of K [8].
A non-zero R-module M is called a rational extension of an R-submodule N of M, if for all \( m_1,m_2 \in M, m_2 \neq 0 \) there exists an element \( r \in R \) such that \( rm_1,m_2 \neq 0 \) [6].

Proposition 1.3
If M is pseudo-N- quasi-injective module, then every essential extension of M is pseudo-N- quasi-injective module.
Proof
Let $H$ be essential extension $R$-module of $M$, and let $f:N \rightarrow H$ be a monomorphism. Since $M$ is an essential submodule of $H$, then $\overline{M} = \overline{H}$, and hence $f:N \rightarrow \overline{M}$ is a monomorphism. But $M$ is pseudo-$N$-quasi-injective module, hence $f(N) \subseteq M$. Therefore $f(N) \subseteq \overline{M}$. The following corollary is immediate consequence of Proposition 1.3.

Corollary 1.4
If $M$ is pseudo-$N$-quasi-injective module, then $\overline{M}$ is pseudo-$N$-quasi-injective module. Since every rational extension is an essential extension [7], we have the following corollary

Corollary 1.5
If $M$ is pseudo-$N$-quasi-injective module, then every rational extension of $M$ is pseudo-$N$-quasi-injective module.

Proposition 1.6
Let $M$, $N$ and $H$ be an $R$-modules. If $M$ is pseudo-$N$-quasi-injective module and $M$ is pseudo- $N \oplus H$-quasi-injective module, then $M$ is pseudo-

Proof
Let $f:N \oplus H \rightarrow \overline{M}$ be a monomorphism and let $j_1:N \rightarrow N \oplus H, j_2:H \rightarrow N \oplus H$ be the injection homomorphism, then $f \circ j_1:N \rightarrow \overline{M}$ and $f \circ j_2:H \rightarrow \overline{M}$ are monomorphisms. But $M$ is both pseudo-$N$-quasi-injective and pseudo-$H$-quasi-injective module. Therefore $f \circ j_1(N) \subseteq M$ and $f \circ j_2(H) \subseteq M$. So $f(N \oplus H) \subseteq M$. Therefore $M$ is pseudo-$N \oplus H$-quasi-injective module.

Corollary 1.7
If $M$ is an $R$-module and $N_1, N_2, \ldots, N_n$ be an $R$-modules, such that $M$ is pseudo-$N_i$-quasi-injective module for each $i=1,2,\ldots,n$, then $M$ is pseudo-$\bigoplus_{i=1}^{n} N_i$-quasi-injective module.

Note:
Let $M$ be an $R$-module, and $N$ be a submodule of $M$, then it is not necessary that $M$ is pseudo-$N$-quasi-injective module as it shown in the following example.

Example 1.8
Then, $Z$-module $Z$ is pseudo-$Z$-quasi-injective. Let $N=3Z$ be a submodule of the $Z$-module $Z$. We claim that $Z$ is not pseudo-$3Z$-quasi-injective module. For if $f:3Z \rightarrow Z$ be a monomorphism, $f(3Z) \subseteq Z$, but $f(3Z) \not\subseteq Z$. Therefore $Z$ is not pseudo-$3Z$-quasi-injective module. However under certain condition on a submodule of $M$, we could treat the above case. First we need to recall the following definitions. A submodule $K$ of an $R$-module $M$ is called pseudo-stable submodule if $f(K) \subseteq K$ for each monomorphism $f:K \rightarrow M$. $M$ is called a fully pseudo stable if each submodule of $M$ is pseudo stable [1].

Proposition 1.9
Let $M$ be an $R$-module, and $N$ is a submodule of $M$. If $N$ is pseudo-stable submodule of $M$, then $M$ is pseudo-$N$-quasi-injective.

Proof
Let $g:N \rightarrow \overline{M}$ be a monomorphism. Since $N$ is pseudo-stable submodule of $\overline{M}$, then $g(N) \subseteq \overline{M}$, and hence $M$ is pseudo-$N$-quasi-injective.

Corollary 1.10
Let $N$ be a submodule of an $R$-module $M$. If $\overline{M}$ is fully pseudo stable $R$-module, then $M$ is pseudo-$N$-quasi-injective. In particular $M$ is pseudo-$M$-quasi-injective.

Corollary 1.11
Let $M$ be fully pseudo-stable module over Noetherian ring, and $N$ is submodule of $M$ then $M$ is pseudo-$N$-quasi-injective.

Proof
Since $M$ is fully pseudo stable $R$-module over Noetherian ring, then $E(M)$ is fully pseudo stable injective envelop of $M$ by [1,Th.2.15 ch.2 ]. Since $\overline{M}$ is a submodule of $E(M)$, then $\overline{M}$ is a pseudo-stable $R$-module [1]. Hence the proof followed by corollary 1.10. Recall that an $R$-module $M$ is terse if every distinct submodules of it are not isomorphic [12]. It is well known that terse module and fully pseudo-stable module are equivalent [1,Prop.2.11 ch2], we have another consequence of Prop.1.9.

Corollary 1.12
Let $N$ be a submodule of an $R$-module $M$, if $\overline{M}$ is terse module, then $M$ is pseudo-$N$-quasi-injective $R$-module. In particular $M$ is pseudo-$M$-quasi-injective $R$-module. Recall that an $R$-module $M$ is Q-module if every submodule of $M$ is quasi-injective [2].

Corollary 1.13
Let $M$ be uniform Q-module over Noetherian ring. If $N$ is a submodule of $M$, then $M$ is pseudo-$N$-quasi-injective module.

Proof
Since $M$ is uniform Q-module, then $\overline{M}$ be fully pseudo stable $R$-module by [1,Th.1.4 ch3]. Now $M$ is fully pseudo-stable submodule over Noetherian ring. Hence the proof followed by cor.1.12.

Note
The class of pseudo-$N$-quasi-injective modules is not closed under submodules. For example $Q$ as $Z$-module is pseudo-$Q$-quasi-injective, and $Z$ is not pseudo-$Q$-quasi-injective submodule.

But it turns out that the class of pseudo-$N$-quasi-injective modules is closed under direct summands. The following proposition shows the case.

Proposition 1.14
A direct summand of pseudo-$N$-quasi-injective module is also pseudo-$N$-quasi-injective for any $R$-module $N$. 


Proof
Suppose that \( M = R \oplus H \) be pseudo-N-quasi-injective R-module. Let \( f : M \to N \) and \( g : N \to \overline{M} \) be homomorphisms. Define \( h : N \to \overline{K} \oplus \overline{H} \) by \( h(n) = ((f(n), g(n))) \) for all \( n \) in \( N \). Clearly, \( h \) is well-defined homomorphism. Let \( i : \overline{K} \oplus \overline{H} \to \overline{M} \) be the inclusion monomorphism, then \( i \circ h : N \to \overline{M} \) is a monomorphism. Since \( M \) is pseudo-N-quasi-injective, then \( i \circ h(N) = h(N) \subseteq M \). but \( h(N) = (f(N), g(N)) \). Therefore \( h(N) = (f(N), g(N)) \subseteq M = K \oplus H \) which implies that \( f(N) \subseteq K \) and \( g(N) \subseteq H \). Hence each of \( K \) and \( H \) are pseudo-N-quasi-injective.

2- Pseudo-Weakly-N-Quasi-injective modules
In this section we introduce the concept of Pseudo-Weakly-N-Quasi-injective module as a generalization of the concept of weakly-N-injective module, and study the basic properties and give examples, characterizations of this concept, and study the relations of Pseudo-Weakly-N-Quasi-injective modules with other know modules.

Definition 2.1
Let \( M \) and \( N \) be two \( R \)-modules, \( M \) is called Pseudo-weakly-N-quasi-injective module if for each monomorphism \( f : N \to \overline{M} \), there exists a submodule \( X \) of \( \overline{M} \) such that \( f(N) \subseteq X \cong M \).

Examples and Remarks 2.2
1. Every weakly-N-injective \( R \)-module is pseudo-weakly-N-quasi-injective module, but the converse is not true as the following example shows:
   - \( Z_2 = \{0, 1\} \) is a \( Z \)-module is pseudo-weakly-Z-quasi-injective module; however \( Z_2 \) is not weakly-Z-injective. For if \( f : Z \to E(Z_2) = Z_{\cong} \) defined by \( f(n) = n/2 + Z \) is a homomorphism for all \( n \in Z \), it is clear that \( f(Z) = Z_{\cong} \) which is not embedded in \( Z_2 \).
2. Every quasi-injective \( R \)-module is pseudo-weakly-N-quasi-injective module for each \( R \)-module \( N \).
3. Every pseudo-injective \( R \)-module is pseudo-weakly-N-quasi-injective module for each \( R \)-module \( N \).
4. \( Z \) as a \( Z \)-module is not pseudo-weakly-Q-quasi-injective module.
5. Every pseudo-N-quasi-injective \( R \)-module is pseudo-weakly-N-quasi-injective \( R \)-module. But the converse is not true.
   - The \( Z \)-module \( Z \) is pseudo-weakly-2Z-quasi-injective, since if \( f : 2Z \to Z \) defined by \( f(2n) = 2n/3 \) for each \( n \in Z \) is a monomorphism. We take \( X = (2/3) \) the submodule of \( Q \) generated by \( 2/3 \). We get \( f(2Z) = (2/3) \cong Z \). However \( Z \) is not pseudo-2Z-quasi-injective since \( f(2Z) \not\subset Z \).
   - The following propositions give some properties of pseudo-weakly-N-quasi-injective modules. Before we give the next proposition we recall the following definition.
   - An \( R \)-submodule \( H \) of an \( R \)-module \( M \) is called fully invariant if \( f(H) \subseteq H \) for all \( f \in \text{End}_R(M) \).[12]

Proposition 2.3
Let \( N_1 \) and \( N_2 \) be two submodules of an \( R \)-module \( M \), such that \( N_1 \subseteq N_2 \) and \( N_2 \) is fully-invariant submodule of \( \overline{M} \). If \( M \) is pseudo-weakly-\( N_2 \)-quasi-injective \( R \)-module, then \( M \) is pseudo-weakly-\( N_1 \)-quasi-injective module.

Proof
Let \( f : N_2 \to \overline{M} \) be a monomorphism, and consider the following diagram

\[
\begin{array}{ccc}
N_2 & \xrightarrow{g_1} & N_1 & \xrightarrow{g_2} & \overline{M} \\
\downarrow{f} & & \downarrow{h} & & \\
\overline{M} & & & & \\
\end{array}
\]

Where \( g_1, g_2 \) are the inclusion homomorphisms, since \( \overline{M} \) is pseudo-injective \( R \)-module, so there exist an \( R \)-homomorphism \( h : \overline{M} \to M \) such that \( h \circ g_2 \circ g_1 = f \). But \( M \) is pseudo-weakly-\( N_2 \)-quasi-injective \( R \)-module and \( g_2 : N_2 \to \overline{M} \) is a monomorphism, so there exist a submodule \( X \) of \( \overline{M} \) such that \( g_2(N_2) = N_2 \subseteq X \cong M \).

Then \( f(N_1) = h \circ g_2 \circ g_1(N_2) = h(N_1) \subseteq N_1 \) (since \( N_1 \) is a fully-invariant submodule of \( \overline{M} \)). Therefore \( f(N_1) \subseteq N_2 \subseteq X \cong M \).

That is \( f(N_1) \subseteq X \cong M \). Therefore \( M \) is pseudo-weakly-\( N_1 \)-quasi-injective \( R \)-module. \( \Box \)

Corollary 2.4
Let \( N_1 \) and \( N_2 \) are two submodules of an \( R \)-module \( M \) such that \( N_1 \subseteq N_2 \) and \( N_2 \) is fully invariant submodule of \( \overline{M} \). If \( M \) is pseudo-weakly-\( N_2 \)-quasi-injective \( R \)-module, then \( M \) is pseudo-weakly-\( N_1 \cap N_2 \)-quasi-injective \( R \)-module.

Since the intersection of finite collection of fully-invariant submodules of an \( R \)-module \( M \) is again fully-invariant submodule [4] we have the following corollary as a consequence of proposition 2.3.

Corollary 2.5
Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be finite family of submodules of an \( R \)-module \( M \) such that \( \lambda_i \) is fully-invariant submodule of \( \overline{M} \) for all \( i = 1, 2, \ldots, n \). If \( M \) is pseudo-weakly-\( N_1 \lambda_i \)-quasi-injective \( R \)-module for all \( i = 1, 2, \ldots, n \), then \( M \) is pseudo-weakly-\( N_1 \cap N_2 \lambda_i \)-quasi-injective \( R \)-module.

Remark 2.6
A direct summand of pseudo-weakly-N-quasi-injective \( R \)-module is not pseudo-weakly-N-quasi-injective \( R \)-module for any \( R \)-module \( N \), as the following example shows.

Let \( M = Z \oplus Q \) and \( N = Q \) and \( R = Z \) we claim that \( Z \oplus Q \) is pseudo-weakly-Q-quasi-injective.

Let \( f : Z \rightarrow Z \oplus Q = Q \oplus Q \) be a monomorphism.

Therefore \( f(Q) \cong 0 \oplus K_1 \) or \( f(Q) \cong K_2 \oplus 0 \) where \( K_1 \) and \( K_2 \) are submodules of \( Q \).
Let \( M \) be a monomorphism. Let \( X \cong M = \mathbb{Z} \oplus Q \). Hence \( M \) is a monomorphism. Let \( f \circ i \). Since \( f \) is an isomorphism, there exists a submodule \( X \) of \( \mathbb{Z} \oplus Q \) such that \( f(X) \cong M = \mathbb{Z} \oplus Q \). Therefore \( X \) is a submodule of \( \mathbb{Z} \oplus Q \). That is \( f_1 : X \to M \) is the inclusion homomorphism. Consider the following diagram:

\[
\begin{array}{ccc}
K & \xrightarrow{i_1} & N \\
\downarrow{f} & & \downarrow{g} \\
\mathbb{M} & & \\
\end{array}
\]

Where \( i_1 : K \to N \) and \( i_2 : N \to \mathbb{M} \) are the inclusion homomorphisms. If \( \mathbb{M} \) is quasi-injective R-module, then there exists a homomorphism \( g : \mathbb{M} \to \mathbb{M} \) such that \( g \circ i_2 \circ i_1 = f \), that is \( g = f \) so \( g \) is a monomorphism also. Since \( g \) is a monomorphism, also, since \( \mathbb{M} \) is quasi-injective R-module so there exist a submodule \( X \) of \( \mathbb{M} \) such that \( h(N) \subseteq X \cong M \). But \( f(K) \subseteq X \cong M \). Hence \( M \) is pseudo-weakly-Q-quasi-injective R-module. \( \Theta \)

**Proposition 2.7**

Let \( M \) be an R-module, and \( N \) be a submodule of \( \mathbb{M} \). If \( M \) is pseudo-weakly-Q-quasi-injective R-module, then \( M \) is pseudo-weakly-Q-quasi-injective R-module for each submodule \( K \) of \( N \).

**Proof**

Let \( f : K \to \mathbb{M} \) be a monomorphism. Consider the following diagram:

\[
\begin{array}{ccc}
K & \xrightarrow{i_1} & N \\
\downarrow{f} & & \downarrow{g} \\
\mathbb{M} & & \\
\end{array}
\]

Where \( i_1 : K \to N \) and \( i_2 : N \to \mathbb{M} \) are the inclusion homomorphisms. Since \( \mathbb{M} \) is pseudo-injective R-module, then there exists a homomorphism \( g : \mathbb{M} \to \mathbb{M} \) such that \( g \circ i_2 \circ i_1 = f \), that is \( g = f \) so \( g \) is a monomorphism also. Let \( h = g \upharpoonright N : N \to \mathbb{M} \). \( h \) is a monomorphism, also, since \( M \) is pseudo-weakly-Q-quasi-injective R-module so there exists a submodule \( X \) of \( \mathbb{M} \) such that \( h(N) \subseteq X \cong M \). But \( f(K) \subseteq X \cong M \). Hence \( M \) is pseudo-weakly-Q-quasi-injective R-module. \( \Theta \)

**Corollary 2.8**

Let \( M \) be an R-module, and \( N, L \) are submodules of \( \mathbb{M} \). If \( M \) is pseudo-weakly-Q-quasi-injective R-module, then \( M \) is pseudo-weakly-N quasi-injective R-module. In the next proposition we show that the class of pseudo-weakly-Q-quasi-injective R-module is closed under essential extension.

**Proposition 2.9**

Let \( M \) and \( N \) be two R-modules such that \( K \) is an essential extension of \( M \). If \( M \) is pseudo-weakly-Q quasi-injective R-module, then \( M \) is also pseudo-weakly-Q quasi-injective R-module.

**Proof**

Let \( f : N \to \overline{K} \) be a monomorphism. Since \( K \) is an essential extension of \( M \) then \( \overline{M} = \overline{K} \). Hence \( f : N \to \overline{M} \) is a monomorphism. But \( M \) is pseudo-weakly-Q quasi-injective, therefore there exists a submodule \( X \) of \( \overline{M} \) such that \( f(N) \subseteq X \cong M \). Let \( g : X \to M \) be an isomorphism. Since \( X \) is a submodule of \( \overline{M} = \overline{K} \), then \( X \) is submodule of \( \overline{K} \). That is \( f_2 : X \to \overline{K} \) is the inclusion homomorphism. Consider the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & \overline{M} \\
\downarrow{f_1} & & \downarrow{h} \\
\overline{K} & & \\
\end{array}
\]

Where \( f_2 : \overline{M} \to \overline{K} \) and \( f_3 : \overline{K} \to \overline{K} \) are inclusion homomorphisms. Since \( \overline{K} \) is pseudo-injective R-module, then there exists a homomorphism \( h_1 : \overline{K} \to \overline{K} \) such that \( h_1 \circ f_3 \circ g = f_1 \). We claim that \( \ker h_1 = \{0\} \) Let \( 0 \neq k_1 \in \overline{K} \) and \( h_1(k_1) = 0 \). Since \( M \) is an essential in \( K \) and \( K \) is an essential in \( \overline{K} \) then \( M \) is an essential in \( \overline{K} \). Therefore there exists \( 0 \neq r \in R \) such that \( 0 \neq rk_1 \in M \) and since \( g \) is an epimorphism, there exists \( x \in X \) such that \( g(x) = rk_1 \). Hence \( r_kx \) is a contradiction, therefore \( k_1 = 0 \), and hence \( h_1 \) is a monomorphism. Let \( h_2 = h_1 \upharpoonright K : K \to \overline{K} \) be a monomorphism. Then \( h_2 \circ f_3 \circ g = f_1 \) and hence \( f_2(X) = X \subseteq h_2(K) \cong K \). Therefore \( f(N) \subseteq X \subseteq h_2(K) \) which mean that \( K \) is pseudo-weakly-Q quasi-injective. \( \Theta \)

The next theorem gives an interesting characterization of pseudo-weakly-Q quasi-injective modules.

**Theorem 2.10**

Let \( M \) and \( N \) be two R-modules. Then \( M \) is pseudo-weakly-Q quasi-injective if and only if for every monomorphism \( f : N \to \overline{M} \) there exists an R-monomorphism \( h : N \to \overline{M} \) and R-monomorphism \( g : M \to \overline{M} \) such that \( g \circ h = f \).

**Proof**

Assume that \( M \) is pseudo-weakly-Q quasi-injective module. Let \( f : N \to \overline{M} \) be a monomorphism. Then there exists a submodule \( X \) of \( \overline{M} \) such that \( f(N) \subseteq X \cong M \). This implies that \( f : N \to X \) is a monomorphism. Let \( \alpha : X \to M \) be an isomorphism we take \( h = \alpha \circ f \). So \( \alpha : X \to M \) is a monomorphism. Let \( g = \inc \circ \alpha^{-1} \circ (\alpha \circ f) \). Now \( g \circ h = (\inc \circ \alpha^{-1} \circ (\alpha \circ f)) \circ h = \inc \circ f = f \). Which prove the only if part.

To prove the if part: let \( f : N \to \overline{M} \) be a monomorphism, then by our hypothesis there exists a monomorphism \( h : N \to M \) and a monomorphism \( g : M \to \overline{M} \) such that \( f = g \circ h \). We take \( X = g(M) \). Then \( X \) is a submodule of \( \overline{M} \) and \( X \cong M \). Moreover \( f(N) = g(h(N)) \subseteq g(M) = X \cong M \). Therefore \( M \) is pseudo-weakly-Q quasi-injective module. \( \Theta \)

The next proposition explains the behavior of pseudo-weakly-Q quasi-injective module under isomorphism.
Proposition 2.11
Let $M$, $N$ and $H$ be an $R$-modules. If $M$ is pseudo-
weakly-$N$-quasi-injective and $N \cong H$ then $M$ is pseudo-
weakly-$H$-quasi-injective module.

Proof
Let $f: H \to \bar{M}$ be a monomorphism and $g: N \to H$ be an
isomorphism. Then $f \circ g: N \to \bar{M}$ is a monomorphism.
But $M$ is pseudo-weakly-$N$-injective, so there exists a submodule $X$ of $\bar{M}$ such that
$f \circ g(N) \subseteq X \cong M$. Hence $f(H) \subseteq X \cong M$ and so $M$ is pseudo-weakly-$H$-quasi-
injective.

Proposition 2.12
Let $M_1$, $M_2$ and $N$ be $R$-modules, such that $M_1 \cong M_2$. If $M_2$
is pseudo-weakly-$N$-quasi-injective, then $M_1$ is pseudo-weakly-$N$-quasi-injective.

Proof
Let $f: N \to \bar{M}_1$ be a monomorphism. Since $M_1 \cong M_2$ then $
\bar{M}_1 = \bar{M}_2$. Hence $f: N \to \bar{M}_1$ is a monomorphism. But
$M_2$ is pseudo-weakly-$N$-quasi-injective $R$-module, therefore there exists a submodule $X$ of $\bar{M}_2$ such that
$f(H) \subseteq X \cong M_1$. That is $g(X)$ is a submodule of $\bar{M}_2$
under an isomorphism $g: \bar{M}_1 \to \bar{M}_2$. So $X \cong g(X)$. Hence
$f(N) \subseteq g(X) \cong X \cong M_1 \cong M_2$ which implies that $M_2$
is pseudo-weakly-$N$-quasi-injective.

Before we give the following proposition, we introduce the following definition.

Definition 2.13
A submodule $N$ of an $R$-module $M$ is called pseudo-
invertible if \( \text{Hom}(M/N, M) = 0 \) for each monomorphism $f: M/N \to M$.

Recall that an $R$-module $M$ is torsion free if $T(M) = \{m \in M: rm = 0 \text{ for some } r \in R\} = \{0\}$.

Proposition 2.14
Let $M$ be a torsion free $R$-module and $N$ be pseudo-
invertible submodule of $M$ such that $M$ is pseudo-
weakly-$M/N$-quasi-injective. Then $N$ is pseudo-
invertible submodule of $\bar{M}$.

Proof
Assume that $N$ is not pseudo-invertible submodule of $\bar{M}$.
That is there exists a non-zero monomorphism $f: \bar{M}/N \to \bar{M}$. Therefore there exists
$\bar{m} = m + N \in \bar{M}/N$ with $m \in \bar{M}$ and $m \notin N$ such that $0 \neq f(m + N) = y$ for some $y \in \bar{M}$. Let $i: M/N \to \bar{M}/N$ be the inclusion homomorphism. Then
$f \circ i: M/N \to \bar{M}$ is a monomorphism. But $M$ is pseudo-
weakly-$M/N$-quasi-injective, so there exists a submodule $X$ of $\bar{M}$ such that $f \circ i(M/N) \subseteq X \cong M$. Let $g: X \to M$ be an isomorphism, then $g \circ f \circ i: M/N \to M$ is a
monomorphism. But since $N$ is pseudo-invertible submodule of $M$, therefore $g \circ f \circ i = 0$. Thus $f \circ i = 0$ and hence $f(M/N) = 0$. But \( f(m + N) \neq 0 \) and $M$ is an essential submodule of $\bar{M}$, therefore, there exists $R \in \bar{M}$ such that $rm \in M$. Hence $rm + N \in M/N$ and $f(rm + N) = 0 = rf(m) + N = \bar{y}$. But $\bar{M}$ is torsion free, so that $r = 0$, which is a contradiction. Therefore $N$ is pseudo-invertible submodule of $\bar{M}$.

Proposition 2.15
Let $M$, $M'$ and $N$ be $R$-modules, such that $\bar{M} \oplus M' = \bar{M} \oplus \bar{M}'$. If $M$ and $M'$ are pseudo weakly-$N$-quasi-injective, then $\bar{M} \oplus \bar{M}'$ is pseudo weakly-$N$-quasi-injective module.

Proof
Let $f: N \to \bar{M} \oplus \bar{M}'$ be a monomorphism, then $f: N \to \bar{M} \oplus \bar{M}'$ be a monomorphism. But $f = (f_1, f_2)$ where $f_1: N \to \bar{M}$ and $f_2: N \to \bar{M}'$ are monomorphism.
Since $M$ is pseudo weakly-$N$-quasi-injective, then there exists a submodule $X$ of $\bar{M}$ such that $f_1(N) \subseteq X \cong M$ and since $M'$ is pseudo weakly-$N$-quasi-injective, then there exist a submodule $Y$ of $\bar{M}'$ such that $f_2(N) \subseteq Y \cong M'$. On other hand $f_2(N) = f_2(N) \oplus 0 \subseteq X \oplus Y \cong M \oplus M'$ and $f(N) = (f_1(N), f_2(N)) \subseteq X \oplus Y \cong M \oplus M'$ which implies that $\bar{M} \oplus \bar{M}'$ is pseudo weakly-$N$-quasi-
injective.

In the following theorem we shall characterize pseudo weakly-$R$-quasi-injective $R$-module $M$.

Theorem 2.16
Let $M$ be an $R$-module. Then $M$ is pseudo weakly-$R$-
quasi-injective if and only if for each element $x$ of $\bar{M}$ with $ann_R(x) = 0$ there exists a submodule $X$ of $\bar{M}$ such that $x \in X \cong M$.

Proof
Assume that $M$ is pseudo weakly-$R$-quasi-injective $R$-
module. Let $x \in \bar{M}$ such that $ann_R(x) = 0$. Define $f: R \to \bar{M}$ by $f(r) = rx$ for all $r \in R$. Clearly $f$ is well-
define monomorphism. But $M$ is pseudo weakly-$R$-quasi-
injective $R$-module. Thus there exists a submodule $X$ of $\bar{M}$ such that $f(R) \subseteq X \cong M$. But $x - 1 \cdot x = f(R)$. Hence $x \in X \cong M$.

Conversely: Suppose that for each element $x \in \bar{M}$ with $ann_R(x) = 0$ there exists $x \in X \cong M$. We have to show that $M$ is pseudo weakly-$R$-quasi-injective. Let $f: R \to \bar{M}$ be a monomorphism. Since $1 \in R$, then $f(1) \in \bar{M}$. Let $x = f(1)$; therefore $\text{ann}_R(x) = \text{ann}_R(f(1)) = f(\text{ann}_R(1)) = f(0) = 0$. Hence there exists a submodule $X$ of $\bar{M}$ such
that \( x \in X \cong M \). To show that \( f(R) \subseteq X \). Let \( b \in f(R) \), then \( b = r f(1) = r x \in X \). Therefore \( f(R) \subseteq X \) and hence \( M \) is pseudo weakly-R-quasi-injective \( R \)-module. \( \theta \)

The next theorem gives a characterization of pseudo weakly-R-quasi-injective \( R \)-module \( R \).

**Theorem 2.17**

Let \( R \) be a ring. Then \( R \) is pseudo weakly-R-quasi-injective \( R \)-module if and only if each element \( a \in R \) with \( ann_R(a) = 0 \) there exists an element \( b \in R \) such that \( \theta \)

\[ a = r b \text{ and } ann_R(b) = 0 \]

**Proof**

Suppose that \( R \) is pseudo weakly-R-quasi-injective \( R \)-module. Let \( a \in R \) such that \( ann_R(a) = 0 \). Define \( f: R \to R \) by \( f(r) = ra \) for all \( r \) in \( R \). It can be easily shown that \( f \) is well-defined \( R \)-monomorphism. Since \( R \) is pseudo weakly-R-quasi-injective \( R \)-module, then there exists a submodule \( X \) of \( R \) such that \( f(R) \subseteq X \cong R \). Clearly \( f(R) = ra \), and hence \( Ra \subseteq X \). which implies that

\[ a = 1.a \in X \]. Let \( g: R \to X \) be an isomorphism. So there exists an element \( c \in R \) such that \( a = g(c) \). Hence \( a = g(c) \in f(R) \subseteq X \). Therefore \( \theta \)

\[ a \in R \]

It is left to show that \( \theta \)

\[ ann_R(a) = 0 \text{ and } ann_R(b) = 0 \]

then \( r/b = 0 \) and hence \( 0 = rf(1) = g(r \cdot 1) = g(r) \) ipilse that \( r/0 \). Hence \( ann_R(b) = 0 \).

Conversely:

\[ \theta \]

Let \( f: R \to R \) be a monomorphism. Then \( f(1) \subseteq R \). Let \( a = f(1) \), then \( \theta \)

\[ ann(a) = ann(f(1)) = f(ann(1)) = f(0) = 0 \]

Therefore there exists an elements \( b \in R \) such that \( \theta \)

\[ a = rb \text{ and } ann_R(b) = 0 \]

Thus \( \theta \)

\[ X \subseteq R \]

Moreover \( \theta \)

\[ f(R) = f(f(1)) \subseteq f(ann(1)) = f(0) = 0 \]

Therefore \( \theta \)

\[ f(R) \subseteq X \cong R \] Which proved that \( R \) is pseudo weakly-R-quasi-injective \( R \)-module. \( \theta \)

References