TENSOR PRODUCT OF CONTINUOUS OPERATOR

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Abstract
In this paper we prove that some properties of tensor product and we show that if \( A_1, A_2 \) are 0-adjoint then \( A_1 \otimes I + I \otimes A_2 \) is normal. Also we prove that a continuous operator is invariant under tensor product.

Introduction
Let \( H \) be an infinite dimensional separable complex Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and let \( \mathcal{B}(H) \) be the algebra of all bounded linear operators on \( H \), given \( \mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}(H) \), the tensor product \( \mathcal{A}_1 \otimes \mathcal{A}_2 \) on the Hilbert space \( H \otimes H \) has been considered variously by many of authors (see [2],[3],[5],[6],[7]). When \( \mathcal{A}_1 \otimes \mathcal{A}_2 \) is defined as follows
\[
\langle \mathcal{A}_1 \otimes \mathcal{A}_2 \left( x_1 \otimes y_1 \right), \left( x_2 \otimes y_2 \right) \rangle = \langle \mathcal{A}_1 x_1, x_2 \rangle \langle \mathcal{A}_2 y_1, y_2 \rangle
\]
The operation of taking tensor product \( \mathcal{A}_1 \otimes \mathcal{A}_2 \) preserves many properties of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \in \mathcal{B}(\mathcal{H}) \) but by no means all of them. Thus, whereas the binormal property is invariant under tensor product, the \( * \)-paranormal property is not [9] again, whereas \( \mathcal{A}_1 \otimes \mathcal{A}_2 \) is posinormal if and only if \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \in \mathcal{B}(\mathcal{H}) \) are [9] and is similarly for \( \mathcal{U} \)-operator, pseudo normal, subnormal and normaloid operators [3],[9],[10]. It was shown in [9] that paranormal is not invariant under tensor product.

In this section we prove some properties of tensor product.

Proposition
If \( A \geq B \geq 0 \) and \( C \geq D \geq 0 \) then
\[
A \otimes C \geq B \otimes D \geq 0
\]
Proof:
Since \( A \geq B \geq 0 \) then
\[
\langle Ax, x \rangle \geq \langle Bx, x \rangle \quad \forall x \in H
\]
And since \( C \geq D \geq 0 \) then
\[
\langle Cx, x \rangle \geq \langle Dx, x \rangle \quad \forall x \in H
\]
\[
\langle Ax, x \rangle \langle Cx, x \rangle \geq \langle Bx, x \rangle \langle Dx, x \rangle
\]
\[
A \otimes C \left( x \otimes x \right) \langle x \otimes x \rangle \geq B \otimes D \left( x \otimes x \right) \langle x \otimes x \rangle
\]
\[
\langle A \otimes C - B \otimes D \left( x \otimes x \right), \left( x \otimes x \right) \rangle \geq 0
\]
\[
\forall x \otimes x \in H \otimes H
\]
then
\[
A \otimes C \geq B \otimes D \geq 0
\]
\[
\begin{align*}
\left( B^2 x, x \right) & \left( A^p x_1, x_1 \right) \left( B^2 x_2, x_2 \right) \geq 0 \\
\left( B^2 x, x \right) & \left( B^p x_1, x_1 \right) \left( B^2 x_2, x_2 \right) \geq 0 \\
\left( B^2 \otimes A^p \otimes B^2 \right) \left( x \otimes x_1 \otimes x_2 \right) \left( x \otimes x_1 \otimes x_2 \right) & \geq 0 \\
\left( B^2 \otimes B^p \otimes B^2 \right) \left( x \otimes x_1 \otimes x_2 \right) \left( x \otimes x_1 \otimes x_2 \right) & \geq 0 \\
\left( B^2 \otimes A^p \otimes B^2 \otimes B^2 \right) & \left( x \otimes x_1 \otimes x_2 \right) \left( x \otimes x_1 \otimes x_2 \right) \geq 0
\end{align*}
\]

hence \( \left\langle r \frac{C^2}{A^p \otimes B^2} \right\rangle \geq \left\langle r \frac{C^2}{A^p \otimes C^2} \right\rangle \)

Similarly \( \left\langle r \frac{A^2 \otimes A^p \otimes A^2}{B^2 \otimes B^p \otimes B^2} \right\rangle \geq \left\langle r \frac{A^2 \otimes B^p \otimes A^2}{B^2 \otimes B^p \otimes A^2} \right\rangle \)

**Proposition**

If \( A \geq C \geq B \geq 0 \) then for each \( r \geq 0, q \geq 1, p \geq 0 \)

\[
\left\langle r \frac{C^2 \otimes A^p \otimes C^2}{q} \right\rangle \geq \left\langle r \frac{C^2 \otimes C^p \otimes C^2}{q} \right\rangle \geq \left\langle r \frac{C^2 \otimes B^p \otimes C^2}{q} \right\rangle \geq 0
\]

**Proof:**

\[\Rightarrow\] since \( A \geq C \geq B \geq 0 \) then \( \langle Ax, x \rangle \geq \langle Cx, x \rangle \geq \langle Bx, x \rangle \geq 0 \) and \( \langle A^p x, x \rangle \geq \langle C^p x, x \rangle \geq \langle B^p x, x \rangle \geq 0 \) \( \forall x \in H \)

\[
\begin{align*}
\left( C^2 \otimes A^p \otimes C^2 \right) \left( x \otimes x_1 \otimes x_2 \right) & \left( x \otimes x_1 \otimes x_2 \right) \geq 0 \\
\left( C^2 \otimes A^p \otimes A^2 \right) \left( x \otimes x_1 \otimes x_2 \right) & \left( x \otimes x_1 \otimes x_2 \right) \geq 0
\end{align*}
\]

\[\langle C^2 \otimes A^p \otimes C^2 \left( x_1 \otimes x \otimes x_2 \right) \left( x_1 \otimes x \otimes x_2 \right) \rangle \geq 0 \]

The following example to show that how to find the Log-operator on a Hilbert space

**Example**

Let \( H \) be a Hilbert space and let \( A, B \) be positive operator matrices such that \( B = \begin{pmatrix} e^2 & 0 \\ 0 & e \end{pmatrix} \) and \( A = \begin{pmatrix} e^4 & 0 \\ 0 & e^{1/2} \end{pmatrix} \) where

\[ V = \frac{1}{\sqrt{5}} \begin{pmatrix} \sqrt{3} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix} \] (unitary), then we have

\[ \log(B) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \] and \( \log(A) = \begin{pmatrix} 2 & \sqrt{3/2} \\ \sqrt{3/2} & 1/2 \end{pmatrix} \)

and \( \log(A) - \log(B) \geq 0 \).

**Theorem**

Let \( A, B, C \) be a positive operator if \( \log(A) \geq \log(C) \geq \log(B) \geq 0 \) then

\[ \langle C^2 \otimes A^p \otimes C^2 \left( x_1 \otimes x \otimes x_2 \right) \langle x_1 \otimes x \otimes x_2 \rangle \rangle \geq 0 \]
\[
\text{Log} \left( C^2 \right) \otimes \text{Log}(A^p) \otimes \text{Log} \left( C^2 \right) \geq 0
\]

\[
\text{Log} \left( C^2 \right) \otimes \text{Log}(C^p) \otimes \text{Log} \left( C^2 \right) \geq 0
\]

\[
\text{Log} \left( C^2 \right) \otimes \text{Log}(B^p) \otimes \text{Log} \left( C^2 \right) \geq 0
\]

**Proof:**

Since \( \text{Log}(A) \geq \text{Log}(C) \geq \text{Log}(B) \) then

\[
\text{Log}(A) + \ldots + \text{Log}(A) \geq \text{Log}(C) + \ldots \geq \text{Log}(B)
\]

\[
\text{Log}(B) + \ldots + \text{Log}(B) \]

\[
p\text{Log}(A) \geq p\text{Log}(C) \geq p\text{Log}(B)
\]

\[
\text{Log}(A^p) \geq \text{Log}(C^p) \geq \text{Log}(B^p)
\]

\[
\langle \text{Log}(A^p) x_1, x_i \rangle \geq \langle \text{Log}(C^p) x_1, x_i \rangle \geq \langle \text{Log}(B^p) x_1, x_i \rangle \]

\[
\text{Log} \left( C^2 \right) x_i, x_i \rangle \text{Log}(A^p) x_1, x_i \rangle \text{Log} \left( C^2 \right) x_2, x_2 \rangle \geq 0
\]

\[
\langle \text{Log} \left( C^2 \right) \otimes \text{Log}(A^p) \otimes \text{Log} \left( C^2 \right) \]

\[
(x \otimes x_1 \otimes x_2) \right) (x \otimes x_1 \otimes x_2) \rangle \geq 0
\]

\[
\langle \text{Log} \left( C^2 \right) \otimes \text{Log}(C^p) \otimes \text{Log} \left( C^2 \right) \]

\[
(x \otimes x_1 \otimes x_2) \right) (x \otimes x_1 \otimes x_2) \rangle \geq 0
\]

\[
\langle \text{Log} \left( C^2 \right) \otimes \text{Log}(B^p) \otimes \text{Log} \left( C^2 \right) \]

\[
(x \otimes x_1 \otimes x_2) \right) (x \otimes x_1 \otimes x_2) \rangle \geq 0
\]

**Theorem**

If \( A_1 \otimes I + I \otimes A_2 \) is \( \theta \)-adjoint then \( A_1, A_2 \) are \( \theta \)-adjoint.

**Proof:**

Since \( A_1 \otimes I + I \otimes A_2 \) is \( \theta \)-adjoint then

\[
A_1^* \otimes I + I \otimes A_2^* = e^{i\theta}(A_1 \otimes I + I \otimes A_2)
\]

\[
A_1^* \otimes I + I \otimes A_2^* = \theta e^{i\theta} A_1 \otimes I - I \otimes e^{i\theta} A_2 = 0
\]

\[
A_1^*, I, e^{i\theta} A_1 \] are linear independent
then \( I = A_1^* = e^{i\theta} A_1 = 0 \) see [2] hence \( I = 0 \)
contradiction . then \( A_1^*, I, e^{i\theta} A_1 \) are linear dependent hence \( A_1^* = \text{re}^{i\theta} A_1 \), \( I = e^{i\theta} \)
and \( A_2^*, I, e^{i\theta} A_2 \) are linear independent then
then \( I = A_2^* = e^{i\theta} A_2 = 0 \) hence \( I = 0 \) contradiction .
Theorem
If \( A_1, A_2 \) are ad-join then \( A_1 \otimes I + I \otimes A_2 \) is normal.

Proof:
Since \( A_1, A_2 \) are ad-join then
\[
A_1^* = e^{i\theta}A_1 \quad \text{and} \quad A_2^* = e^{i\theta}A_2, \quad \theta_1, \theta_2 \in \mathbb{R}
\]
\[
A_1A_2^* \otimes I + I \otimes A_2A_1^* - A_1^*A_2^* \otimes A_1 \otimes I = I \otimes A_2^*A_1^* - e^{-i\theta}A_1^*e^{i\theta}A_1 \otimes I + I \otimes e^{-i\theta}A_2^*e^{i\theta}A_2
\]
\[
A_1^*A_2^* \otimes I - I \otimes A_2^*A_1 = 0
\]

hence \( A_1 \otimes I + I \otimes A_2 \) is normal.

In this section we may assume that \( A \not\in B(H) \).
If \( A : H \to H \) let \( x \in H, A \) is continuous at \( x \) incase \( x_n \to x \) impels \( Ax_n \to Ax \) [1]

Proposition
If \( A_i, i = 1, \ldots, n \) are operator and \( A_1 \otimes A_2 \otimes \cdots \otimes A_n \) on Hilbert space \( H_1 \otimes H_2 \otimes \cdots \otimes H_n, A_1 \otimes A_2 \otimes \cdots \otimes A_n \) is continuous operator if each \( A_i, i = 1, \ldots, n \) is continuous operator.

Proof:
By induction it is enough to show that \( A_1 \otimes A_2 \) is continuous if \( A_1 \) and \( A_2 \) are
If \( \|x_n \otimes y_m - x \otimes y\| = 0 \) we prove that
\[
\|A_1x_n \otimes A_2y_m - A_1x \otimes A_2y\| = 0
\]

since \( A_1 \) is continuous if \( x_n \to x \) then
\( A_1x_n \to A_1x \)
and \( A_2 \) is continuous if \( y_m \to y \) then
\( A_2y_m \to A_2y \)
\[
\|A_1x_n \otimes A_2y_m - A_1x \otimes A_2y\| = \|A_1x_n \otimes A_2y_m - A_1x_n \otimes A_2y\| + A_1x_n \otimes (A_2y_m - A_2y)
\]
\[
A_1x_n \otimes (A_2y_m - A_2y) + (A_1x_n - A_1x) \otimes A_2y = 0
\]
then \( A_1 \otimes A_2 \) is continuous.

References


الخلاصة

نيرهن في هذا البحث بعض خواص الجداء النسولي ونوضح ان المؤثر المستمر يحافظ على خواصته تحت تأثير الجداء النسولي.