Strongly C-Compactness

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Abstract
In this paper, we define another type of compactness which is called "strongly c-compactness ". Also, we study some properties of this type of compactness and the relationships with compactness, strongly compactness and c-compactness.

1. Introduction and Preliminaries
A topological space \((X, \tau)\) is said to be c-compact space if for each closed set \(A \subseteq X\), each open cover of \(A\) contains a finite subfamily \(W\) such that \(\{\text{cl } v : v \in W\}\) covers \(A\), [1].

Mashhour et.al.[2] introduced preopen sets, [A subset \(A\) of space \(X\) is said to be preopen set if \(A \subseteq \text{int (cl}(A))\)]. Obviously each open set in \((X, \tau)\) is preopen, not conversely.

Also, they defined the following concepts:

i. A is called a preclosed set iff \((X – A)\) is preopen set.

ii. The intersection of all preclosed sets contain \(A\) is called the preclosure of \(A\) and denoted by \(\text{pre cl} A\)

iii. The prederived set of \(A\) is the set of all elements \(x\) of \(X\) satisfies the condition, that for every preopen set \(V\) contains \(x\), implies \(V \setminus \{x\} \cap A \neq \emptyset\).

Also, they proved some properties, as (the preclosure of a set \(A\) is a preclosed set) and (preclosure \((B) = B\) iff \(B\) is preclosed set).

Pre-open sets are discussed in [3], [4].

Ganster [5] has shown that the family of all preopen sets in \(X\) (PO(X)) is a topology on \(X\) if closure \(G\) is open and \(\{x\}\) is preopen for each \(x \in \text{interior } F\) where \(X = F \cup G\).

A space \((X, \tau)\) is called strongly compact if every preopen cover of \((X, \tau)\) admits a finite subcover.

Proposition (1.1),[1]:
Every compact space is c-compact.

Remark (1.2):
The implication in proposition (1.2) is not reversible, for example: A space \((N, \tau)\) where, \(\tau = \{U_n = \{1,2,...,n\} : n \in N\} \cup \{\emptyset\}\) is c-compact which is not compact.

Definition (1.3),[9]:
A topological space \((X, \tau)\) is said to be a \(T_3\)-space iff it is regular and \(T_1\)-space.

Proposition (1.4),[1]:
A \(T_3\)-c-compact space is compact.

Proposition (1.5), [6], [7], [8]:
Every strongly compact space is compact.

Remark (1.6):
The opposite direction of proposition (1.5) may be false, for example:

Let \(X = [0,1]\) as a subspace of \((R, \tau_a)\). Clearly, \(X\) is compact, but not strongly compact space, since the preopen cover
\[
C = \{(0,\frac{1}{2}) \cup \{\frac{1}{n} : n \in N\} \cup \{\frac{1}{3},1\}\cup \{\left(\frac{1}{n} - r_n, \frac{1}{n} + r_n\right) \mid r_n = \frac{1}{2(n+1)^2} \land n > 2\}\}
\]
has no finite subcover.
Proposition (1.7), [7], [8]:
If the set of accumulation points of X is finite, then X is strongly compact space, whenever it is compact space.

In proposition (1.8) and remark (1.9) below we discuss the relationship between strongly and c-compact spaces.

Proposition (1.8):
Every strongly compact space is c-compact.

Proof:
Follows directly from propositions (1.5) and (1.1).

Remark (1.9):
The opposite direction of proposition (1.8) may be false, see the example in remark (1.2), (N,τ) is c-compact space which is not strongly compact, since \{1, n\} \ n \in N \} is a preopen cover for N which has no finite subcover.

In the following proposition we give some conditions to make the opposite direction of proposition (1.8) true.

Proposition (1.10):
A T_3-c-compact space X is strongly compact, whenever the set of accumulation points of X is finite.

Proof:
Follows directly from propositions (1.4) and (1.7).

2. Strongly c-compactness:
In this section we shall introduce the concept of strongly c-compactness and the relationships among compact, c-compact, strongly compact and strongly c-compact spaces are examined.

Definition (2.1):
A topological space X is said to be "strongly c-compact space" if for each preclosed set A ⊆ X, each family \{V_a: a \in \Lambda\} of preopen sets in X and covering A there is a finite subfamily W such that \{pre-cl V_a: V_a \in W\} covers A.

Proposition (2.2):
A strongly compact space is strongly c-compact.

Proof:
Clear.

Remark (2.3):
The opposite direction of proposition (2.2) need not be true, see the example of remark (1.2), (N,τ) is strongly c-compact which is not strongly compact.

Proposition (2.4):
A T_3 strongly c-compact space is strongly compact.

Proof:
Let X be a T_3 strongly c-compact space. If X is not strongly compact, then there is a preopen cover \{u_a: a \in \Lambda\} for X which has no finite subcover. Since X is strongly c-compact space, then there is a finite subfamily W of the preopen cover \{u_a: a \in \Lambda\} such that X = \bigcup_{i=1}^n \{pre-cl u_a, u_a \in W\}. This means, there is x \in X, x \in pre-cl u_a, but x \notin u_{a_i} for some i = 1, 2, ..., n. Implies x \in pre-derived u_{a_i}. Since X is T_3-space, then \{x\} is a closed set and x \notin u_{a_i}, implies y \notin \{x\} \forall y \in u_{a_i}. Since X is regular, then there are two open sets V_y and V'_y such that y \in V_y and \{x\} \subseteq V'_y and V_y \cap V'_y = \emptyset for each y \in u_{a_i}.

Therefore \{V'_y\}_{y \in u_{a_i}} is an open cover for \{x\}. But \{x\} is compact, then there is \{V'_{y_1}, V'_{y_2}, ..., V'_{y_n}\} covers \{x\}.

Let V' = \bigcup_{i=1}^n V'_{y_i}, then V' is an open set contains x. Let V = \bigcup_{y \in u_{a_i}} V_y, then V is an open set contains u_{a_i}, and V \cap V' = \emptyset. Since, every open set is a preopen, then V and V' are preopen sets and x \in V', u_{a_i} \subseteq V and V \cap V' = \emptyset. Therefore, x \notin pre-derived u_{a_i}, which is a contradiction. Then X is a strongly compact space.

Corollary (2.5):
A T_3 strongly c-compact space is compact.

Proof:
Follows from propositions (2.4) and (1.5).
**Remark (2.6):**

In general a strongly c-compact space need not be compact, see the example of remark (1.2), \((\mathbb{N},\tau_1)\) is strongly c-compact space which is not compact.

On the other hand, a compact space may not be strongly c-compact, for example: The compact space \((\mathbb{N},\tau_I)\), where \(\tau_I\) is the indiscrete topology on \(\mathbb{N}\), is strongly c-compact since \(\{\{n\}|n\in\mathbb{N}\}\) preopen cover for \(\mathbb{N}\), which has no finite subfamily \(W\) such that \(\{\text{pre-cl } u|u \in W\}\) covers \(\mathbb{N}\), since \(\text{pre-cl}\{n\} = \{n\} \forall n \in \mathbb{N}\).

In the following proposition we add a condition to make any compact space strongly c-compact space

**Proposition (2.7):**

If the set of accumulation points of \(X\) is finite, \(X\) is strongly c-compact space whenever it is a compact space.

**Proof:**

Follows from propositions (1.7) and (2.2).

**Proposition (2.8):**

A strongly c-compact space is c-compact.

**Proof:**

Let \(X\) be a strongly c-compact space, to prove it is c-compact. If not, then there is a closed set \(A \subseteq X\) and an open cover \(\{u_\alpha: \alpha \in \Lambda\}\) for \(A\), such that \(A \neq \bigcup_{i=1}^{n} \text{cl } u_\alpha, \forall n \in \mathbb{N}\). Since, every open set is preopen, then \(\{u_\alpha: \alpha \in \Lambda\}\) is a preopen cover for \(A\), then there is a finite subfamily \(\{u_\alpha_i: i = 1,2,...,m\}\) such that \(\{\text{pre-cl } u_\alpha_i: i=1,2,...,m\}\) covers \(A\).

This means, there exists \(x \in A\) such that \(x \in \text{pre-cl } u_\alpha_i\) and \(x \notin \text{cl } u_\alpha_i\), for some \(i=1,2,...,m\).

Since \(x \notin \text{cl } u_\alpha_i\), implies \(x \notin u_\alpha_i\), but \(x \in \text{pre-cl } u_\alpha_i\) then \(x \in \text{pre-derived } u_\alpha_i\).

On the other hand, since \(x \notin \text{cl } u_\alpha_i\), implies \(x \notin u_\alpha_i\) and \(x \notin \text{derived } u_\alpha_i\). Therefore, there exists an open set \(V\) such that \(x \in V\) and \(V \cap u_\alpha_i = \emptyset\).

Now, we get a preopen set \(V\) such that \(x \in V\) and \(V \cap u_\alpha_i = \emptyset\), implies \(x \notin \text{pre-derived } u_\alpha_i\), which is a contradiction.

Therefore \(X\) is c-compact whenever it is strongly c-compact space. ■

**Remark (2.9):**

A c-compact space need not be strongly c-compact. As the space \((\mathbb{N},\tau_I)\).

In the following proposition we add some conditions to make c-compact space to be strongly c-compact.

**Proposition (2.10):**

In a T_{3\frac{1}{2}}\text{-space } (X,\tau), if the set of accumulation points of \(X\) is finite, then the concepts of c-compactness and strongly c-compactness are coincident.

**Proof:**

Follows from propositions (1.4) and (2.7).

The following diagram shows the relationships among the different types of compactness we studied in this section.
3. Certain Fundamental Properties of Strongly c-Compact Space

In this section we shall discuss some properties of strongly c-compact spaces.

**Remark (3.1):**

Strongly c-compactness is not a hereditary property, as the following example shows;

Let \( X = \mathbb{N} \cup \{0\}, \)
\( \tau = \mathcal{P}(\mathbb{N}) \cup \{\emptyset \cup \{n \in \mathbb{N} \mid 0 \in H \land X - H \text{ is finite}\}. \)

Now, \( X \) is a strongly compact space, implies \( X \) is strongly c-compact space (by proposition (2.2)). But, \( \mathbb{N} \subseteq X \) not strongly c-compact since \( \{\{n\} \mid \{n\} \in \mathbb{N} \} \) is a preopen cover for \( \mathbb{N} \) which has no finite subfamily \( W \) such that \( \{\text{pre-cl}(n) :\{n\} \in W\} \) cover \( \mathbb{N} \).

**Remark (3.2):**

The continuous image of a strongly c-compact space need not be strongly c-compact. For example;

Let \( f : (\mathbb{N}, \tau) \to (\mathbb{N}, \tau_i) \) such that \( f(x) = x \forall x \in \mathbb{N} \) where \( \tau_i = \{U_n \mid U_n = \{1, 2, \ldots, n\} \mid n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\}. \) Then, \( f \) is a continuous function and \( (\mathbb{N}, \tau_i) \) is strongly c-compact space, but \( (\mathbb{N}, \tau) \) is not strongly c-compact.

**Definition (3.3), [10]:**

Let \( f : (X, \tau) \to (Y, \tau') \) be any function, \( f \) is said to be a preirresolute function, if and only if the inverse image of any preopen set in \( Y \) is a preopen set in \( X. \)

**Remark (3.4) [10]:**

A function \( f : (X, \tau) \to (Y, \tau') \) is a preirresolute iff the inverse image of any preclosed set in \( Y \) is a preclosed set in \( X. \)

**Lemma (3.5):**

A function \( f : (X, \tau) \to (Y, \tau') \) is a preirresolusle if and only if \( \text{pre-cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)) \forall B \subseteq Y. \)

**Proof:**

Necessity, let \( f : (X, \tau) \to (Y, \tau') \) be a preirresolute function and let \( B \subseteq Y. \)

Since \( B \subseteq \text{pre-cl}(B) \) then \( f^{-1}(B) \subseteq f^{-1}(\text{pre-cl}(B)) \). implies \( \text{pre-cl}(f^{-1}(B)) \subseteq \text{pre-cl}(f^{-1}(\text{pre-cl}(B))). \) Since \( f \) is preirresolute function and \( \text{pre-cl}(B) \) is preclosed set in \( Y \), then \( f^{-1}(\text{pre-cl}(B)) \) is preclosed set in \( X. \) So, \( \text{pre-cl}(f^{-1}(\text{pre-cl}(B))) = f^{-1}(\text{pre-cl}(B)). \)

Therefore, \( \text{pre-cl}(f^{-1}(B)) \subseteq f^{-1}(\text{pre-cl}(B)). \)

Sufficiency, suppose \( \text{pre-cl}(f^{-1}(B)) \subseteq f^{-1}(\text{pre-cl}(B)) \forall B \subseteq Y. \) To prove \( f \) is preirresolute function.

We must prove that if \( A \) is preclosed set in \( Y, \) then \( f^{-1}(A) \) is preclosed set in \( X. \)

Which means : we must prove that \( f^{-1}(A) = \text{pre-cl}(f^{-1}(A)) \). It is clear that \( f^{-1}(A) \subseteq \text{pre-cl}(f^{-1}(A)) \forall A \subseteq Y. \)

Now, to prove \( \text{pre-cl}(f^{-1}(A)) \subseteq f^{-1}(A), \) Since \( A \) is preclosed set in \( Y, \) then \( \text{pre-cl}(A) = A \) and since \( \text{pre-cl}(f^{-1}(A)) \subseteq f^{-1}(\text{pre-cl}(A)). \)

Implies, \( \text{pre-cl}(f^{-1}(A)) \subseteq f^{-1}(A). \)

Therefore, \( \text{pre-cl}(f^{-1}(A)) = f^{-1}(A) \) and \( f^{-1}(A) \) is a preclosed set in \( X. \) So \( f \) is preirresolute function. \( \blacksquare \)

**Proposition (3.6):**

The preirresolute image of a strongly c-compact space is a strongly c-compact.

**Proof:**

Let \( f : (X, \tau) \to (Y, \tau') \) be a preirresoluste onto function and let \( X \) be a strongly c-compact space. To prove \( Y \) is strongly c-compact space.

Let \( A \) be a preclosed subset of \( Y, \{u_\alpha : \alpha \in \Lambda\} \) be a \( \tau' \)-preopen cover for \( A. \) Since \( f \) is a preirresoluste function, implies \( f^{-1}(u_\alpha) : \alpha \in \Lambda \) is a \( \tau \)-preopen cover for a preclosed set \( f^{-1}(A) \subseteq X \) and since \( X \) is strongly c-compact space, then there is a finite family \( \{u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_n}\} \) such that \( \{\text{pre-cl}(f^{-1}(u_{\alpha_i}) : i = 1, 2, \ldots, n\} \) covers \( f^{-1}(A) \). So \( \{\text{pre-cl}(f^{-1}(u_{\alpha_i})) : i = 1, 2, \ldots, n\} \) covers \( A. \) In virtue of lemma (3.5), \( \{f^{-1}(\text{pre-cl}(u_{\alpha_i})) : i = 1, 2, \ldots, n\} \) covers \( A \) and since \( f \) is onto, then \( \{\text{pre-cl}(u_{\alpha_i}) : i = 1, 2, \ldots, n\} \) covers \( A. \) Hence, \( Y \) is strongly c-compact space. \( \blacksquare \)

**Proposition (3.7), [10]:**

Every homeomorphism function is a preirresoluste function.

**Corollary (3.8):**

A strongly c-compactness is a topological property.

**Proof:**

In virtue of proposition (3.7), then proposition (3.6) is applicable. \( \blacksquare \)
4. Conclusion and Recommendations:

Our conclusions in this paper, that a strongly c-compact space is c-compact space but not strongly compact space and not compact space. So we have to strive to put another type of compactness which lies between strongly compactness and c-compactness.

For future works, we shall study $\alpha$-c-compactness, semi-$\alpha$-c-compactness, semi-p-compactness and semi-p-c-compactness.

References