On New Algebraic Systems

Mehsin Jabel Atteya* Dalal Ibraheem Ressan*

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Abstract:
The purpose of this paper is to give some results , theorems and corollaries concerning new algebraic systems flower , garden and farm with accustomed algebraic systems groupoid, group and ring.

Keywords: Groupoid , Group , Ring ,Semiflower , Flower , Garden ,Farm , Lahhamian Group.

Introduction:
Many authors work for development and find out some mathematics systems. Al-Lahham [1] one of them , he was introduced new algebraic system flower , garden and farm ,he gave some results as a finite farm ( S, * ) such that S>2, is not a field . Al-Lahham introduced type of groups is called a Lahhamian group, during his work depended on [2] and [3] . We can say he find some relations between his systems with another accustomed algebraic systems.

Preliminaries :
Throughout this paper S denote to a groupoid is an order pair (S, * )where S is a non empty set and * is a binary operation on S [2] unless explicity stated .A binary operation * on non empty set S is ATL if for all a,b,c ∈S ,a*(b*c)=c*(b*a).Asemiflower is an order pair (S, * ) where S is a non empty set and * is an ATL binary operation on S .Aflower is an order pair (S, * ) where S is a non empty set and * is a binary operation on S satisfying the following axioms :

(i )a*(b*c)=c*(b*a) for all a,b,c ∈S (ATL law).

(ii )There exists an element e in S such that a*e =a for all a ∈S(e is a right identity of S).

(iii )a*a=e for all a ∈S.A group (S, * ) is called Lahhamian group if a*a=e for all a ∈S , where e the identity of S.

A garden is a triple (S, *, ∘ ) ,where S is non empty set has at least two elements , * and ∘ are two binary operations on S, ( ∘ :S×S→S)and S’ =S- {0},(0 is the right identity in (S, *)) ,satisfies the following axioms:

(i ) (S, *) is a flower .( ii ) ∘ is ATL (i.e.a ∘ (b ∘ c)=c ∘ (b ∘ a)for all a,b,c ∈S’).

(iii )The distributive law , (a*b) ∘ c=(a ∘ c) * (b ∘ c) for all a,b ∈S, c ∈S’ , holds in S. A farm ( S, *, ∘) is a garden such that ( S’, ∘) is a flower.A function p[ λ ]from a flower S to S is a right (left)translation of S if p(a*b)= a(P(b)) [ λ (ab)]= λ (a)b for all a,b ∈ S .The operation o is composition of functions.

Now ,we will mention some results which we need to a chieve our work which apperared in[1]

Lemma 1 [Lemma 4]
Let (S, *, ∘) be a garden ,then for all a ∈S’, 0 ∘ a=0.

Lemma 2 [Theorem6]
A flower (S, *) is a Lahhamian group if and only if it is commutative .

Lemma 3 [Theorem4]
Let \( S \) be a flower, then every right translation of \( S \) commutes with every left translation of \( S \).

**Proposition 5** [**Proposition 2**]
If \((S, \ast)\) is an abelian group, then \( S \) with the new binary operation \( \ast \) defined as follows:
\[ a \ast b = a \cdot b \] for all \( a, b \in S \) is a flower.

**Lemma 4** [**Lemma 2**]
If \((S, \ast)\) is a flower then for all \( a, b \in S, a \ast b = e \ast (b \ast a) \).

**Proposition 6** [**Proposition 1**]
If \((S, \ast)\) is a flower then \( (b \ast c) \ast a = (b \ast a) \ast c \) for all \( a, b, c \in S \).

**Proposition 7** [**Proposition 3**]
If \( S \) is a Lahhamian group then \( S \) is a flower.

**Proposition 8** [**Theorem 8**]
Let \((S, \ast)\) be a flower, then the following conditions are equivalent:
(i) \( S \) is a Lahhamian group.
(ii) \( S \) has an identity.
(iii) \( S \) is commutative.
(iv) \( S \) is associative.

It is our aim in this paper to give some new results concerning new algebraic systems.

**The Main Results:**

**Theorem 1**
Let \((S, \ast)\) be a non-empty set such that \( \ast \) is idempotent on \( S \), \( \ast \) satisfy ATL law and left cancellation law hold in \( S \), then \((S, \ast)\) is a flower.

**Proof:** We have \( a \ast a = a \), \( a \in S \), \( \ast \) satisfy ATL law. Then we must prove
(i) \( a \ast e = a, a \in S \).
(ii) \( a \ast a = e, a \in S \). According to our relation, we have
\[ a \ast a = e, a \in S \]. Thus \( e \) is a right identity element of \( S \).

**Remark 2**
In **Theorem 1** \((S, \ast)\) is a semiflower.

**Proposition 3**
If \((S, \ast)\) is a flower then the following are true for all \( a, b, c \in S \).
1. \( S \) is closed under \( \ast \).
2. \( (e \ast a) \ast (a \ast e) = e \) (\( e \) is a right identity element of \( S \)).
3. \( (b \ast a) \ast a = b \)

**Proof:**
(1) Since \( \ast \) is binary operation, then for all \( a, b, c \in S \), we obtain \( a \ast b \in S \), replacing \( b \) by \( a \), we have \( a \ast a \in S \), by axiom (iii), we get \( e \in S \), \( e \) is a right identity element of \( S \).

(2) We have \( (e \ast a) \ast (a \ast e) = e \). Let \( x = (e \ast a) \) and \( y = (a \ast e) \), then \( x \ast y = e \). We set \( x = e \ast x \) and \( y = y \ast e \), then
\[ (e \ast x) \ast (y \ast e) = (e \ast (e \ast a)) \ast ((a \ast e) \ast e) \]. By ATL law, we get
\[ = (a \ast (e \ast b)) \ast (e \ast a) \]. By Proposition 6, we obtain
\[ = b \ast (e \ast (a \ast e)). By ATL law, we obtain \]
\[ = b \ast (e \ast a) \ast (a \ast e) \]. By Part 2, then
\[ = b \ast e = b \].

**Theorem 4**
Let \((S, \ast)\) be a flower, if \( a \ast x = y \ast b \) then \( x = y \) for all \( a, b, x, y \in S \).

**Proof:** We have \( a \ast x = y \ast b \) for all \( a, b, x, y \in S \). Then
\( a \ast (a \ast x) = a \ast (y \ast b) \) for all \( a, b, x, y \in S \).

By ATL law, we obtain
\[
x \ast (a \ast a) = b \ast (y \ast a)
\]
for all \( a, b, x, y \in S \).

By axiom (ii), we get
\[
x \ast e = b \ast (y \ast a)
\]
for all \( a, b, x, y \in S \).

By axiom (ii), we obtain
\[
x = b \ast (y \ast a)
\]
for all \( a, b, x, y \in S \).

By ATL law, we get
\[
e \ast (x \ast e) = e \ast ((y \ast a) \ast (b \ast e))
\]
for all \( a, b, x, y \in S \).

By axiom (ii), we obtain
\[
(x \ast e) = e \ast ((y \ast a) \ast (b \ast e))
\]
for all \( a, b, x, y \in S \).

By ATL law, we obtain
\[
x = b \ast (e \ast a)
\]
for all \( a, b, x, y \in S \).

By ATL law, we obtain
\[
x = (a \ast y) \ast (e \ast b)
\]
for all \( a, b, x, y \in S \).

Let \( b = e \ast b \), then
\[
x = (a \ast y) \ast (e \ast (e \ast b))
\]
for all \( a, b, x, y \in S \).

By ATL law, we obtain
\[
x = (a \ast y) \ast (b \ast e)
\]
for all \( a, b, x, y \in S \).

Then
\[
x = (a \ast y) \ast b
\]
for all \( a, b, x, y \in S \).

By Proposition 6, we obtain
\[
x = (a \ast y) \ast b
\]
for all \( a, b, x, y \in S \).

Replacing \( b \) by \( e \ast y \), we obtain
\[
x = e \ast (e \ast y)
\]
for all \( x, y \in S \).

By ATL law, we get
\[
x = y
\]
for all \( x, y \in S \).

**Theorem 5**

If \((S, \ast)\) is an abelian group, then \( S \) with the new binary operation \( \ast \) defined as follows: \( a \ast b = a \ast (b \ast c) \ast e \ast (b \ast a) \) for all \( a, b, c \in S \).

Proof: (i) \( c \ast (b \ast a) = e \ast (b \ast a) \) for all \( a, b, c \in S \).

Since \( S \) is belain then
\[
c \ast (b \ast a) = e \ast (b \ast a)
\]
for all \( a, b, c \in S \).

Since \( e \) is identity element of \( S \), we obtain
\[
(a \ast e) \ast (b \ast c) = a \ast (b \ast c)
\]
for all \( a, b, c \in S \).

Since \( e \) is identity element of \( S \), we get
\[
((a \ast e) \ast (b \ast c)) \ast (a \ast e) = a \ast (b \ast c)
\]
for all \( a, b, c \in S \).

Then
\[
((a \ast e) \ast (b \ast c)) \ast (a \ast e) = a \ast (b \ast c)
\]
for all \( a, b, c \in S \).

Then
\[
= a \ast (b \ast c)
\]
for all \( a, b, c \in S \).

Then \( \ast \) satisfying ATL law.

(ii) Satisfying from our hypothesis \( e \) is a right identity element for \( \ast \) on \( S \).

(iii) \( a \ast a = a \ast a = e \) for all \( a \in S \). Then \((S, \ast)\) is a flower.

**Proposition 6**

Let \((S, \ast)\) be a flower such that \( a \ast b = b \ast 1 \) for all \( a, b \in S \), then \( S \) is Lahhamian group, where \((S, \ast)\) is a group.

Proof: We have \( a \ast b = a \ast b \ast 1 \) for all \( a, b \in S \). Then we must prove \( a \ast b = b \ast a \) for all \( a, b \in S \). Then we suppose \( a \ast b \neq b \ast a \) for some \( a, b \in S \). Then
\[
ab \ast 1 \neq ba \ast 1
\]
for some \( a, b \in S \). Then
\[
a1 \neq 1a
\]
for some \( a, b \in S \). Since \( e \) is a right identity element of \( S \), we obtain
\[
ab \ast 1 \neq ba \ast 1
\]
for some \( a, b \in S \). Then replacing \( b \) by \( a \), we get
\[
aa \ast a \neq a \ast a
\]
for some \( a \in S \). Then
\[
a \neq a
\]
for some \( a \in S \). This lead to contradiction.

Then \((S, \ast)\) is a commutative, by Lemma 2, \((S, \ast)\) is Lahhamian group.

**Theorem 7**

Every flower is commutative.

Proof: Suppose \((S, \ast)\) is a flower, then we have \( a \ast (b \ast c) = (b \ast c) \ast a \) for all \( a, b, c \in S \). Then by ATL law, we obtain
\[
a \ast (b \ast c) = (b \ast c) \ast a
\]
for all \( a, b, c \in S \).
c \ast (b \ast a) = (b \ast c) for all a, b, c \in S.
Then
e \ast (c \ast (b \ast a)) = (e \ast (a \ast (b \ast c))) for all a, b, c \in S. e is a right identity element of S. By ATL law, we get
(b \ast a) \ast (c \ast e) = (b \ast c) \ast (a \ast e) for all a, b, c \in S. Since e is a right identity, then
(b \ast a) \ast c = (b \ast c) \ast a for all a, b, c \in S.

Replacing a by (b \ast c) and c by (b \ast a), we obtain
(b \ast (b \ast c)) \ast (b \ast a) = (b \ast (b \ast a)) \ast (b \ast c) for all a, b, c \in S.

By ATL law, we get
(c \ast (b \ast b)) \ast (b \ast a) = (a \ast (b \ast b)) \ast (b \ast c) for all a, b, c \in S. Then
(c \ast e) \ast (b \ast a) = (a \ast e) \ast (b \ast c) for all a, b, c \in S.

Replacing a by b, we obtain (c \ast e) \ast (b \ast b) = (b \ast e) \ast (b \ast c) for all b, c \in S. According to axiom (iii), we get
(c \ast e) \ast e = e. Since e is a right identity of S. Then c = e for all c \in S, then c \ast b = e \ast b for all c, b \in S. By Theorem 4, we get b = e for all b \in S. Then
c \ast b = b \ast e for all c, b \in S. Since c = e, then
c \ast b = b \ast c for all c, b \in S. Then (S, \ast) is commutative.

**Corollary 8:**
Let (S, o) be a flower with left and right translation of S, then (S, o) is Lahhamian group.

**Proof:** By Lemma 3, left and right translation are commutative i.e. (S, o) is commutative flower. By Lemma 2, we get (S, o) is Lahhamian group.

**Theorem 9**
The Boolean ring (S, \ast, \diamond) is garden, where (S, \ast) is Lahhamian group.

**Proof:** (i) By Proposition 7, (S, \ast) is flower.
(ii) For all a, b, c \in S, we have a \diamond (b \diamond c), replacing b by c, we obtain
a \diamond (b \diamond c) = a \diamond (c \diamond c). Since c \diamond c = c, c \in S, S is Boolean ring, then

= a \diamond c

By replacing c by a, we obtain

= a \diamond a = a.

Also, we have c \diamond (b \diamond a), by same method, we obtain c \diamond (b \diamond a) = a.

Thus, we obtain a \diamond (b \diamond c) = c \diamond (b \diamond a) for all a, b, c \in S.

i.e. \diamond is ATL law.

**Theorem 10**
Let (S, \ast, \diamond) be a garden, then

1. \((a \diamond b) \ast (c \diamond d) = 0\) for all a, b, c, d \in S.
2. c \ast (b \diamond a) = 0 for all \(c \in S, a, b \in S\).
3. (a \ast b) \diamond (c \ast d) = 0 for all a, b, c, d \in S.

**Proof:**
(i) We have \((a \ast b) \ast (c \ast d)\), replacing a by c and b by d, we obtain
\((a \ast b) \ast (c \ast d) = (c \ast d) \ast (c \ast d) = 0\) for all c, d \in S. (c is a right identity element of (S, \ast)).

(ii) By replacing c by (b \ast a), we get
\(c \ast (b \ast a) = (b \ast a) \ast (b \ast a) = 0\) for all c, d \in S.

(iii) Since o is a right identity of (S, \ast), then have

\(o \ast o = (a \ast a) \ast (a \ast a)\) for all a \in S.

Since o is a right identity of (S, \ast), then for all a \in S,

\(= ((a \ast a) \ast o) \ast (a \ast a)\). By distributive law, we obtain

\(= ((a \ast a) \ast (a \ast a)) \ast (o \ast (a \ast a))\) for all a \in S.

Since o is a right identity of (S, \ast), then

\(=((a \ast a) \ast (a \ast a)) \ast ((a \ast a) \ast (a \ast a))\) for all a \in S.

(4) For all a, b, c, d \in S we have \((a \ast b) \diamond (c \ast d)\). By take a = b, we obtain
\((b \ast b) \diamond (c \ast d) = 0 \diamond (c \ast d)\). Since o is a right identity of (S, \ast), then

\(=((c \ast d) \ast (c \ast d)) \diamond (c \ast d)\). By distributive law, we get

\(=((c \ast d) \ast (c \ast d)) \ast ((c \ast d) \ast (c \ast d))\)

= 0 (o is a right identity of (S, \ast)).
Theorem 11
In a garden $(S, \ast, \circ)$ for all $a,b \in S$, $c \in S^\ast$, then $c \circ (a \ast b)=0$.

Proof: We have $c \circ (a \ast b)$, then $(c \ast o)$ $\circ (a \ast b)$ for all $a,b \in S$, $c \in S^\ast$. By distributive law, we get $(c \ast o) \circ (a \ast b) = (c \circ (a \ast b)) \circ (o \ast (a \ast b))$ for all $a,b \in S$, $c \in S^\ast$.

Let $a=b$, we obtain

$= (c \circ (a \ast a)) \ast (o \circ (a \ast a))$ for all $a \in S$, $c \in S^\ast$.

Since $(S, \ast)$ is a flower then by axiom (iii), we get

$= (c \circ o) \circ (o \circ o)$. By distributive law, we obtain

$= (c \ast o) \circ o$ (o is right identity of $(S, \ast)$). By Lemma 1, we get $=(c \ast (o \ast y)) \circ (o \circ y)$ for all $c$, $y \in S^\ast$. By distributive law, we obtain $=(c \circ (o \circ y)) \ast ((o \circ y) \circ (c \circ y)$ for all $c, y \in S^\ast$. Then

$=(c \circ (o \circ y)) \ast o$ for all $c$, $y \in S^\ast$. Since $o$ is a right identity of $(S, \ast)$, then

$= c \circ (o \circ y)$ for all $c, y \in S^\ast$. By Lemma 1, we obtain

$= c \circ o$ for all $c \in S^\ast$. By Theorem 10(3), we obtain

$= c \circ (o \circ o)$ for all $c \in S^\ast$. By ATL law, we get

$= o \circ (o \circ o)$ for all $c \in S^\ast$. By Lemma 1, we obtain

$= o \circ o$. By Theorem 10(3), we get

$= 0$

As a corollary of Theorem 11, we can write

Corollary 12
Let $(S, \ast, \circ)$ be a garden, then for all $c \in S^\ast$, $c \circ o = 0$.

Proof: We have $c \circ o$, for all $c \in S^\ast$, by Theorem 10(3), we obtain

$c \circ o = c \circ (o \circ o)$ for all $c \in S^\ast$. By ATL law, we get

$= o \circ (o \circ o)$ for all $c \in S^\ast$. By Lemma 1, we obtain

$= o \circ o$. Again by Theorem 10(3), we get

$= 0$

Proposition 13
In a farm $(S, \ast, \circ)$ for all $a,b,c \in S$, then $c \circ (a \ast b) = 0$.

Proof: We can write the proof of our proposition by same method in Theorem 11.

From precedence Theorem 10, Theorem 11 and Corollary 12, we can write the following remark.

Remark 14
Let $(S, \ast, \circ)$ be a garden, then

$(a \circ b) \ast (c \circ d) = (a \ast b) \circ (c \ast d) = c \ast (b \circ a) = c \circ (a \ast b) = c \circ o = o \circ o = o$.

Theorem 15
If $(S, \ast, \circ)$ be a ring, then $S$ with the new operation $\circ$ and $\ast$ defined as follows:

$a \ast b = a \cdot b^{-1}$ for all $a,b \in S$.

$a \circ b = a$ for all $a,b \in S$. Is a garden, where $(S, \ast)$ is a belian group.

Proof: (i) By Proposition 5, $(S, \ast, \circ)$ is a flower.

(ii) We have $c \circ (b \circ a)$ according to defined of $\circ$, then $c \circ (b \circ a) = c \circ b$ for all $a,b,c \in S$. Since $(a \circ b = a)$, then

$= (c \circ a) \circ (b \circ a)$ for all $a,b,c \in S$.

Replacing $a$ by $c$ and $c$ by $a$, we obtain

$= (a \circ c) \circ (b \circ c)$ for all $a,b,c \in S$. Since $(a \circ c = a)$ according to defined of $\circ$, we obtain

$= a \circ (b \circ c)$ for all $a,b,c \in S$.

Thus $\circ$ is ATL law.

(iii) We have $(a \ast b) \circ c = a \ast b$ for all $a,b \in S$. According to defined of $\circ$, we obtain

$= (a \circ c) \ast (b \circ c)$ for all $a,b,c \in S$. Thus the distributive law hold in $S$.

Then $(S, \ast, \circ)$ is a garden.

Theorem 16
If $(S, \ast, \circ)$ be a ring, then $S$ with new operation $\circ$ and $\ast$ defined as follows:

$a \ast b = a \cdot b^{-1}$ for all $a,b \in S$ and $a \circ b = a$ for all $a,b \in S$. Is a farm, where $(S, \ast)$ is a belian group.
**Proof:** According to Theorem 15, we have 
\((S, \ast)\) is a flower and the distributive law hold in \(S\), therefore, we must prove 
\((S', \diamond)\) is a flower, then
(i) Form Theorem 15, \(\diamond\) is ATL law of \(S'\).
(ii) According to \(a \diamond b = a\) for all \(a, b \in S'\), there exists an element \(e\) in \(S'\) such that \(a \diamond e = a\) for all \(a \in S'\), i.e. \(e\) is a right identity element of \(S'\).
(iii) According to \(a \diamond b = a\) for all \(a, b \in S'\), then \(a \diamond a = a\) for all \(a \in S'\), therefore,
\[ a \diamond a = a \diamond (a \diamond e). \]
By ATL law, we get \(= e \diamond (a \diamond a)\) for all \(a \in S'\). According to defined \(\diamond\), we obtain \(= e \diamond a\) for all \(a \in S'\). According to defined \(\diamond\), we obtain \(= e\). Then \((S', \diamond)\) is a flower.
Thus, we obtain \((S', \ast, \diamond)\) is a farm.
As a corollary of Theorem 8 and Proposition 8, we write the following theorem.

**Theorem 17**
If \((S, \ast)\) is a flower. Then
(i) \(S\) is a Lahhamian group.
(ii) \(S\) is has an identity.
(iii) \(S\) is commutative.
(iv) \(S\) is a associative.

The following conjecture appeared in [1] with out proof, we give the proof.

**Al-Lahham Conjecture 18**
If \((S, \ast)\) is a groupoid with a right identity element \(e\), such that \(x \ast x = e\) for all \(x \in S\), then for all elements \(a, b, c \in S:\)
\[ a \ast (b \ast c) = c \ast (b \ast a) \] if and only if
\[ (a \ast b) \ast c = (a \ast c) \ast b. \]

**Proof:** For all \(a, b, c \in S\), we have \(a \ast (b \ast c) = c \ast (b \ast a)\). Then we will deal with \((S, \ast)\) as a flower. By Proposition 6, we get
\[ (a \ast c) \ast c = (a \ast c) \ast b. \]
Also, we have for all \(a, b, c \in S\), such that \((a \ast b) \ast c = (a \ast c) \ast b\).
Since \(x \ast x = e\) for all \(x \in S\).
Then \((S, \ast)\) is a Lahhamian group, by Proposition 7, \((S, \ast)\) is flower, then \(a \ast (b \ast c) = c \ast (b \ast a)\) is satisfy.

**References:**

**حول بنى جبرية حديثة**

 Mahsen Jibril Usama

* دلال إبراهيم رسن

* الجامعة المستنصرية - كلية التربية - قسم الرياضيات

**الخلاصة:**

الغرض الرئيسي من هذا البحث أعطاء بعض النتائج مبهرات وتمييزيات ولاحظات بخصوص هذه البنى الجبرية الحديثة، الزهرة والحييقة والمزرعة مع بنى جبرية مألوفة تصف الزمرة والزمرة والحلقة.