On Representation of Monomial Groups

S.A. Bedaiwi and A.A. Hajim
Department of Mathematics, College of Science, Al-Mustansiryah University, Baghdad-Iraq.

Abstract
Taketa shows that all monomial groups (commonly written as M-groups) are solvable. Gajendragadkar gives the notion of \( \pi \)-factorable character. We show that an irreducible character of an M-group is primitive if it is \( \pi \)-factorable. Issacs proves that product of two monomial characters is a monomial. We extend this fact to include any finite number of monomial characters consequently we prove that any product of finite number of M-groups is an M-group. We show that any group of order 45 is an M-group and for any group \( G \), the factor group \( G/\text{G}^' \) is an M-group.

Keywords: Representation theory, Monomial groups, \( \pi \)-factorable characters.

1. Introduction
The essential body of representation theory has been constructed by Richard Brauer (1901-1977). His processors; Frobenius, Burnside and Schur, gave the grand task to which character theory could make a central contribution, that is, the complete classification of finite simple groups [1], [2], [13].

T. Okuyanta [12] proved that if G is an M-group and \( P \) is Sylow P-subgroup of G, then \( N_G(P)/P \) is an M-group. I.M. Issacs [7] shows that if H is a Hall subgroup of an M-group then \( N_G(H)/H' \) is also an M-group.

In studying monomial groups it is important to know as much as possible about the primitive characters of its subgroups, since that every character is induced from a primitive character [4].

The following are proved:

- Any irreducible character of monomial group is primitive if it is \( \pi \)-factorable.
- Any finite product of monomial characters is monomial.
- The external direct product of \( n \)-copies of monomial groups is monomial.

2. Characters and M-groups
Character theory was developed by Frobenius in 1896. It provides a powerful tool for proving theorems about finite groups. No non-character theoretic description of the class of M-groups has been found. We use character techniques to gain more information and facts about M-groups.

2.1 Definition [8]:
Let \( \chi \) be a character of G, then \( \chi \) is monomial if \( \chi = \lambda G \) where \( \lambda \) is a linear character of some subgroup of G.

2.2 Definition [7]:
Let G be any group, we denote by \( \text{Irr}(G) \) for the set of all irreducible characters of G.

2.3 Definition [8]:
A group G is an M-group (monomial group) if every \( \chi \in \text{Irr}(G) \) is monomial character.

2.4 Theorem (Taketa) [1]:
Every M-group is solvable

2.5 Theorem [8]:
Every nilpotent group is an M-group.

2.6 Definition [2]:
Let \( \pi = \{p_1, p_2, ..., p_n\} \) be a non-empty set of primes a \( \pi \)-number is a positive integer whose prime divisors belong to \( \pi \). An element of a group is called a \( \pi \)-element if its order is a \( \pi \)-number and if every element of a group is \( \pi \)-element, the group is called \( \pi \)-group.

2.7 Remark:
Let \( \pi \) be a set of primes define \( \pi' \) to be the complement primes of \( \pi \), the \( \pi' \)-number is a positive integer whose prime divisors does not belong to \( \pi \). An element of a group is called a \( \pi' \)-element if its order is a \( \pi' \)-number and if every element of a group is \( \pi' \)-element the group is called \( \pi' \)-group.
2.8 Definition [9]:
Let $G$ be a finite group and let $\pi$ be a nonempty set of primes. Then $G$ is said to be $\pi$-separable if it has normal series each factor of which is either a $\pi$-group or $\pi'$-group.

2.9 Definition [7]:
Let $\chi$ be a character of $G$ and $\det \chi = \lambda$ be the uniquely defined linear character, write $o(\chi) = o(\lambda)$ the order of $\lambda$ as an element of the group of linear characters of $G$ is called the determinantal order of $\chi$.

2.10 Definition [6]:
Let $\chi \in \text{Irr}(G)$, we say that $\chi$ is $\pi$-factorable if there exist $\xi, \eta \in \text{Irr}(G)$ such that $\chi = \xi \eta$.

2.11 Definition [6]:
Irreducible characters whose restriction to every normal subgroup is homogeneous (multiples of an irreducible) are called quasi-primitive.

2.12 Theorem [6]:
Let $G$ be an $M$-group. Then $\chi \in \text{Irr}(G)$ is primitive if it is $\pi$-factorable.

3 Main Results and Applications

3.1 Proposition:
Let $G$ be a $\pi$-separable group, $\chi \in \text{Irr}(G)$ is quasi-primitive. Then the $\pi$-special and $\pi'$-special factors of $\chi$ are quasi-primitive.

Proof:
We can write $\chi = \zeta \eta$ where $\zeta$ is $\pi$-special and $\eta$ is $\pi'$-special. Let $N \triangleleft G$ and let $\alpha$ and $\beta$ be irreducible constituent of $\zeta_N$ and $\eta_N$ respectively, then $\alpha \beta$ is irreducible and is a constituent of $\chi_N$. Since $\chi$ is quasi-primitive it follows that $\alpha \beta$ is $G$-invariant and thus $\alpha$ and $\beta$ are $G$-invariant by the uniqueness of factorization.

3.2 Theorem:
Let $G$ be any group, $\theta \in \text{Irr}(N)$ then $\theta$ is called primitive if it cannot be obtained by inducing any character of proper subgroup.

3.3 Remark:
Any finite product of monomial characters is monomial.

3.4 Proposition:
External direct product of n-copies of monomial group is monomial.

Proof:
Let $G_i$ be monomial group for each $i$, to show that $\prod_{i=1}^{n} G_i$ is monomial. Let $\chi = \prod_{i=1}^{n} \eta_i \in \text{Irr}(\prod_{i=1}^{n} G_i)$ where $\eta_i \in G_i$, since $G_i$ is monomial group for each $i$, by Definition 2.1 $\eta_i$ is monomial character for $G_i$. Therefore, $\chi$ is monomial.
each \( i \), by Remark 3.3 \( \chi = \prod_{i=1}^{n} \eta_i \) is monomial.
therefore \( \prod_{i=1}^{n} G_i \) is monomial group.

3.5 Proposition: Any group of order 45 is an M-group.

Proof:
Let \( G \) be any group of order 45, since 
\[ 45 = 3^2 \cdot 5 \]
\( G \) has a 3-sylow subgroup \( H \) of order 9 and a 5-sylow subgroup \( K \) of order 5. Let \( n \) is the number of the distinct conjugates of \( H \), then \( n = 1 + 3r \) \((r \geq 0)\) and \( n \) divides 45 the only possibility is \( r = 0 \) thus \( n = 1 \) and hence \( H \) is normal in \( G \). Similarly \( K \) is normal in \( G \), we have \( G = HK \). Since 
\[ |HK| = |HK| |H \cap K| = |H||K| = 45 \]
thus \( G \) isomorphic to \( H \times K \) but \( H \) is abelian and \( K \) is cyclic [11] so \( G \) is abelian and hence it is nilpotent therefore by theorem 2.5 it is an M-group.

3.6 Proposition: Let \( G \) be a group and let \( G' \) be the derived subgroup of \( G \), then \( G/G' \) is an M-group.

Proof:
We know that \( G' \) is normal in \( G \), let \( x, y \in G \) then;
\( (xG')(yG') = xy[(x^{-1}y^{-1}xy)G'] = (x^{-1}y^{-1}y^{-1}y^{-1})y^{-1}G' = yG'xG' \)
since \( (x^{-1}y^{-1}y^{-1})y^{-1} \in G' \) thus \( G/G' \) is abelian and hence nilpotent (in fact every abelian group is nilpotent group of class one) therefore it is M-group by theorem 2.5.

3.7 Proposition:
The quotient group \( GL(2, R)/SL(2, R) \) is an M-group.

Proof:
We show that \( SL(2, R) \) is the derived subgroup of \( GL(2, R) \) and by using proposition 3.9 we are done. The mapping \( GL(2, R) \to \) defined by \( x \to \det(x) \) is a homomorphism with kernel \( SL(2, R) \), thus the special linear group is a normal subgroup and \( [GL(2, R)]' \subseteq SL(2, R) \).

Now, the following matrices are the generators of \( SL(2, R) \):

\[
\begin{pmatrix}
1 & r \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
s & 1 \\
1 & 0
\end{pmatrix}, \begin{pmatrix}
t & 0 \\
0 & 1/t
\end{pmatrix}, \begin{pmatrix}
t & 0 \\
1/t & 0
\end{pmatrix}.
\]

Where

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in SL(2, R) \text{ we have two cases; when}
\]

\[
\begin{pmatrix}
a & 0 \\
0 & 1/a
\end{pmatrix}
\]

And the calculation below show that the generators of \( SL(2, R) \) are commutators

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 1/2 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
1/2 & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

And therefore we have:

\( SL(2, R) \subseteq [GL(2, R)]' \)

Acknowledgment
We would like to express our deep gratitude for the referee (s) for the suggested valuable comments.

References