Solution of Some Application of System of Ordinary Initial Value Problems Using Osculatory Interpolation Technique

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Abstract
The aim of this paper is to find a new method for solving a system of linear initial value problems of ordinary differential equation using approximation technique by two-point osculatory interpolation with the fit equal numbers of derivatives at the end points of an interval [0, 1] and compared the results with conventional methods and is shown to be that seems to converge faster and more accurately than the conventional methods.

Key words : Initial value problems, Approximation, Osculatory interpolation

1- Introduction
Systems of ordinary differential equations (ODEs) arise in mathematical models throughout science and engineering. When an explicit condition (or conditions) that a solution must satisfy is specified at one value of the independent variable, usually its lower bound, this is referred to as an initial value problem (IVP) and a system of ordinary differential equations is a system of equations relating several unknown functions y(x) of an independent variable x, some of the derivatives of the y_i(x), and possibly x itself. [1] Initial-value problems for systems of differential equations permeate many areas of mathematics; such problems arise naturally in modelling the evolution of dynamical processes in economics, engineering, and the physical, biological sciences [2]. In this paper we introduce reactor problem which introduced in [3] Kehoe and Butt have studied the kinetics of benzene hydrogenation on a supported Ni/kieselguhr catalyst. In the presence of a large excess of hydrogen, the reaction is pseudo-first-order at temperatures below 200°C with the rate given by

\[-r = P_{H_2}k_0K_0 T \exp\left(\frac{-(Q-E_a)}{R_g T}\right) C_B \text{ mol/(g catalyst s)}\]

where
Rg = gas constant, 1.987 cal/(mole*K)
- Q - E_a = 2700 cal/mole
P_{H_2} = hydrogen partial pressure (torr)
Price and Butt [4] studied this reaction in a tubular reactor. If the reactor is assumed to be isothermal, we can calculate the dimensionless concentration profile of benzene in their reactor given plug flow operation in the absence of inter- and intraphase gradients. Using a typical run,

\[ k_o = 4.22 \text{ mole/(gcat·s·torr)} \]
\[ K_o = 2.63 \times 10^{-6} \text{ cm}^3/(\text{mole·K}) \]
\[ T = \text{absolute temperature (K)} \]
\[ C_B = \text{concentration of benzene (mole/cm}^3{)} \]

And if we now consider the reactor to be adiabatic instead of isothermal, then an energy balance must accompany the material balance. Formulate the system of governing differential equations.

The data of this problem
\[ C_p = 12.17 \times 10^4 \text{ J/(kmole·°C)} \]
\[ -\Delta H_r = 2.09 \times 10^8 \text{ J/kmole} \]
\[ T^* = \frac{T}{T^0}, T^0 = 423K (150^0 \text{ C}) \]

We have system of initial value problem

\[ \frac{dy}{dx} = -0.1744 \exp\left[\frac{3.21}{T^*}\right] y \] (material balance)
\[ \frac{dT^*}{dx} = 0.06984 \exp\left[\frac{3.21}{T^*}\right] y \] (energy balance)
with I.C \( y(0) = 1 \), \( T^*(0) = 1 \)

2- Problem definition

In this section we can explain the way through the application of this system, of initial value problem:

\[ \frac{dy}{dx} = -0.1744 \exp\left[\frac{3.21}{T^*}\right] y \]

\[ \frac{dT^*}{dx} = 0.06984 \exp\left[\frac{3.21}{T^*}\right] y \]

with I.C \( y(0) = 1 \), \( T^*(0) = 1 \)

In this paper we are particularly concerned with fitting function values and derivatives at the two end points of a finite interval, say \([0,1]\), wherein a useful and succinct way of writing a osculatory interpolant \( P_{2n+1}(x) \) of degree \( 2n + 1 \) was given for example by Phillips [5] as:
\[ P_{2n+1}(x)=\sum_{j=0}^{n} \{ y^{(j)}(0) q_{j}(x)+(-1)^{j} y^{(j)}(1) q_{j}(1-x) \} \] ...............(2)

\[ q_{j}(x) = \binom{n+s}{n} x^{s} q_{j}(x)/j! \] ..............(3)

so that (2) with (3) satisfies

\[ y^{(r)}(0)= P_{2n+1}^{(r)}(0) \quad y^{(r)}(1)= P_{2n+1}^{(r)}(1) \quad r=0,1,2,\ldots,n. \]

We can write the equation (2) directly in terms of the Taylor coefficients \( a_{i} \) and \( b_{i} \) about \( x=0 \) and \( x=1 \) respectively, as

\[ P_{2n+1}(x)=\sum_{j=0}^{n} \{ a_{j} Q_{j}(x) + (-1)^{j} b_{j} Q_{j}(1-x) \} \] ..............(4)

The simple idea of this paper is to replace \( y(x) \) in problem (1) by a \( P_{2n+1} \) in equation (3). The first step therefore is to construct the \( P_{2n+1} \). To do this we need the Taylor coefficients of \( y(x) \) and \( T^{*}(x) \) respectively about \( x=0 \)

\[ y(x)= a_{0} + a_{1} x + \sum_{i=2}^{\infty} a_{i} x^{i} \] ............(5a)

Where \( y(0)=a_{0}, \quad y'(0)=a_{1} \quad \ldots \quad y^{(i)}(0)/i! =a_{i} \quad i=2,3,\ldots \)

And

\[ T^{*}(x)= b_{0} + b_{1} x + \sum_{i=2}^{\infty} b_{i} x^{i} \] ............(5b)

Where \( T^{*}(0)=b_{0}, \quad T^{*}'(0)=b_{1} \quad \ldots \quad T^{*^{(i)}}(0)/i! =b_{i} \quad i=2,3,\ldots \)

Also we need the Taylor coefficients of \( y(x) \) and \( T^{*}(x) \) respectively about \( x=1 \)

\[ y(x)= c_{0} + c_{1} (x-1) + \sum_{i=2}^{\infty} c_{i} (x-1)^{i} \] ............(6a)

Where \( y(1)=c_{0}, \quad y'(1)=c_{1} \quad \ldots \quad y^{(i)}(1)/i! =c_{i} \quad i=2,3,\ldots \)

\[ T^{*}(x)= d_{0} + d_{1} (x-1) + \sum_{i=2}^{\infty} d_{i} (x-1)^{i} \] ............(6b)

Where \( T^{*}(1)=d_{0}, \quad T^{*}'(1)=d_{1} \quad \ldots \quad T^{*^{(i)}}(1)/i! =d_{i} \quad i=2,3,\ldots \)

Then we simply insert the series forms in (5a) in to equation (1) and equate coefficients of \( x \) to obtain \( a_{i} \), then derive equation (1) and insert the series in to (5a) and equate coefficients of \( x \) to obtain \( a_{2} \) and soon to obtain \( a_{3}, a_{4} \ldots \) and simply insert the series forms in (6a) in to equation (1) and equate coefficients of \( x-1 \) to obtain \( c_{1} \), then derive equation (1) and insert the series in to (6a) and equate coefficients of \( x \) to obtain \( c_{2} \) and soon to obtain \( c_{3}, c_{4} \ldots \). Then equation (6b) in the same manner to obtain \( d_{2}, d_{3},\ldots \)

The resulting system of equations can be solved to obtain \( (a_{0}, a_{1}, a_{2}) \) for all \( i \geq 2 \).

The notation implies that the coefficients depend only on the indicated unknowns \( a_{0}, b_{0}, c_{0}, d_{0} \).

We note here there are only two variables \( c_{0}, d_{0} \) because all the unknowns in terms of \( a_{0}, b_{0} \) so requires then presence of only two equations.

Now integrate equation (1) to obtain:

\[ c_{0} - a_{0} + \int_{0}^{1} f_{1}(x, y, T^{*}) \, dx = 0 \] ...............(7a)

\[ d_{0} - b_{0} + \int_{0}^{1} f_{2}(x, y, T^{*}) \, dx = 0 \] ...............(7b)
and replacement $P_{2n+1}$, $\tilde{P}_{2n+1}$ of $y, T^*$ in (7a) and (7b) respectively and insert $c_0$ and $d_0$ and $a_i$, $b_i$, $c_i$, $d_i$ in to $P_{2n+1}$, $\tilde{P}_{2n+1}$.

Then solve system of algebraic equation using matlab to obtain $c_0$ and $d_0$ and insert into (4) which represent the solution of (1).

From equations (2), (3) we have the solution when $n=3,4$:

$$P_7=-.6200000e-18X^7-1.110814X^6+3.35442X^5-.5768X^4-.3313X^3+.134693e-2X^2+4.14525*x+.24730108e-3$$

$$P_9=-.821544e-17X^9+.233415X^8-.8651138e-3X^5-.123418e-1X^4+.166667e-1X^3+.155100X^2-.103300X-.30012$$

And

$$\tilde{P}_7=-.122040e-3X^7+.48829e-3X^6-.8651138e-3X^5-.123418e-1X^4+.166667e-1X^3+.155100X^2-.103300X-.30012$$

$$\tilde{P}_9=.19740e-6X^9-.833318e-8X^8+.577163e-X^7+.345225e-3X^6-.832243e-3X^5-.125044e-1X^4+.163267e-1X^3+.1521000X^2-.10010X-.3354$$

It is clear that from table 2, the suggested method is more accurate that the other results and converge faster and easy implementation.

**References**

1- Youdong Lin, Joshua A., Enszer, and Mark A.,(2007), Stadtherr1, Enclosing All Solutions of Two-Point Boundary Value Problems for ODEs

2- Russell L. Herman,(2008), A Second Course in Ordinary Differential Equations of Dynamical Systems and Boundary Value Problems.


The results of solution given in the following table:
Table 1: The result of the methods for n = 3, 4 of example

<table>
<thead>
<tr>
<th>P7</th>
<th>P9</th>
<th>( \tilde{p}_7 )</th>
<th>( \tilde{p}_9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>c0</td>
<td>1.32667921198</td>
<td>1.32668875443</td>
<td>1.32667925438</td>
</tr>
<tr>
<td>d0</td>
<td>-2.6828498769</td>
<td>-2.6833878269</td>
<td>-2.682849108</td>
</tr>
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</table>

Now we give a comparison between the solution of suggested method and solution of other methods in the following table

Table 2: A Comparison between \( P_9 \) and other methods of example

<table>
<thead>
<tr>
<th>X</th>
<th>DVERK, TOL = (-6)</th>
<th>DGEAR, TOL = (-4)</th>
<th>( P_9 ) by using Osculatory interpolation</th>
<th>DVERK, TOL = (-6)</th>
<th>DGEAR, TOL = (-4)</th>
<th>( \tilde{p}_9 ) by using Osculatory interpolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000000000</td>
<td>1.000000</td>
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<td>1.000000000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.700367</td>
<td>0.700468</td>
<td>0.7003665774</td>
<td>1.1199</td>
<td>1.11994</td>
<td>1.1199859943</td>
</tr>
<tr>
<td>0.2</td>
<td>0.529199</td>
<td>0.529298</td>
<td>0.5291928843</td>
<td>1.1889942932</td>
<td>1.18894291</td>
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</tr>
<tr>
<td>0.3</td>
<td>0.4137326433</td>
<td>0.4137326475</td>
<td>1.2349569243</td>
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<tr>
<td>0.4</td>
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<tr>
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<td>0.2456766932</td>
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<td>1.2934506322</td>
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<tr>
<td>0.6</td>
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<td>0.7</td>
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<td>0.2178286938</td>
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<tr>
<td>0.8</td>
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<tr>
<td>0.9</td>
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<tr>
<td>1</td>
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<td>0.1009877429</td>
<td>1.3600221044</td>
<td>1.36002210432</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
حل بعض تطبيقات منظومة من مسائل القيم الابتدائية الاعتيادية باستخدام تقنية الاندراج التماسي

خالد منديل محمد الأبراهيمي
قسم الرياضيات - كلية التربية - جامعة القادسية
استلم البحث في: 25 أيار 2011، قبل البحث في: 7 كانون الأول 2011

الخلاصة
الهدف من هذا البحث هو إيجاد طريقة جديدة لحل منظومة من مسائل القيم الابتدائية المعادلة التفاضلية الاعتيادية، إذ استعملت تقنية التقريب ذا الاندراج التماسي ذي النقاطين التي تتفق فيها الدالة وعدد متساو من النقاط المعروفة عند نقطة نهاية المدة [0, 1] مع البيانات المعطاة. وقررت الطريق المفترضة مع الطرق التقليدية وقد ظهرت النتائج بأن الطريق المفترضة ذو تقارب أسرع و أكثر دقة من الطرق التقليدية.

كلمات مفتاحية: مسائل القيم الابتدائية، التقريب، الاندراج التماسي