The Construction and Reverse Construction of the Complete Arcs in the Projective 3-Space Over Galois Field GF(2)

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Abstract

The main purpose of this work is to find the complete arcs in the projective 3-space over Galois field GF(2), which is denoted by PG(3,2), by two methods and then we compare between the two methods.

Keywords: arcs, secant, quadable.

Introduction, [1,2]

A projective space PG(3,q) over Galois field GF(q), q = p^m, for some prime number p and some integer m, is a 3-dimensional projective space.

Any point in PG(3,q) has the form of a quadable \((x_1, x_2, x_3, x_4)\), where \(x_1, x_2, x_3, x_4\) are elements in GF(q) with the exception of the quadable consisting of four zero elements.

Two quadrables \((x_1, x_2, x_3, x_4)\) and \((y_1, y_2, y_3, y_4)\) represent the same point if there exists \(\lambda\) in GF(q) \(\neq \{0\}\) such that \((x_1, x_2, x_3, x_4) = \lambda (y_1, y_2, y_3, y_4)\), this is denoted by \((x_1, x_2, x_3, x_4) \equiv (y_1, y_2, y_3, y_4)\).

Similarly, any plane in PG(3,q) has the form of a quadable \([x_1, x_2, x_3, x_4]\), where \(x_1, x_2, x_3, x_4\) are elements in GF(q) with the exception of the quadable consisting of four zero elements.

Two quadrables \([x_1, x_2, x_3, x_4]\) and \([y_1, y_2, y_3, y_4]\) represent the same plane if there exists \(\lambda\) in GF(q) \(\neq \{0\}\) such that \([x_1, x_2, x_3, x_4] = \lambda [y_1, y_2, y_3, y_4]\), this is denoted by \([x_1, x_2, x_3, x_4] \equiv [y_1, y_2, y_3, y_4]\).

Also a point \(P(x_1, x_2, x_3, x_4)\) is incident with the plane \(\pi [a_1, a_2, a_3, a_4]\) iff \(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0\).

Every line in PG(3,q) contains \(q + 1\) points and every point is on exactly \(q + 1\) lines. Any plane in PG(3,q) contains exactly \(q^2 + q + 1\) points and \(q^2 + q + 1\) lines. Every point is on \(q^2 + q + 1\) planes and is on \(q^2 + q + 1\) lines.

Moreover PG(3,q) contains exactly \(q^3 + q^2 + q + 1\) points and also contains exactly \(q^3 + q^2 + q + 1\) planes.

Definition 1: [1,3]

A \((k,n)\) – arc A in PG(3,q) is a set of \(k\) points such that at most \(n\) points of which lie in any plane, \(n \geq 3\). \(n\) is called the degree of the \((k,n)\) – arc.
Definition 2: [1,3]

In PG(3,q), if A is any (k,n) – arc, then an (n-secant) of A is a plane \( \pi \) such that \( |\pi \cap A| = n \).

Definition 3: [1,3]

Let \( T_i \) be the total number of the \( i \) – secants of a (k,n) – arc A, then the type of A denoted by \( (T_n, T_{n-1}, \ldots, T_0) \).

Definition 4: [1,3]

Let \( (k_1,n) \) – arc A is of type \( (T_n, \ldots, T_0) \) and \( (k_2,n) \) – arc B is of type \( (S_n, \ldots, S_0) \), then A and B are projectively equivalent iff \( T_i = S_i \).

Definition 5: [1,3]

If a point \( N \) not on a (k,n)-arc A has index i iff there are exactly i(n –secants) of A through \( N \), one can denote the number of points \( N \) of index i by \( C_i \).

Definition 6:

If \( (k,n) \)-arc A is not contained in any \( (k + 1,n) \)-arc, then A is called a complete \( (k,n) \)-arc.

Remark:

From definition 5, it is concluded that the \( (k,n) \)-arc is complete iff \( C_0 = 0 \).

Thus the \( (k,n) \)-arc is complete iff every point of PG(3,q) lies on some n-secant of the \( (k,n) \)-arc.

1- The Construction of Complete \( (k,n) \)-Arcs in PG(3,2)

1.1 The Construction of Complete \( (k,3) \)-arcs in PG(3,2):

PG(3,q) contains 15 points and 15 planes such that each point is on 7 planes and every plane contains 7 points (see table 1).

The set \( A = \{1, 2, 3, 4, 13\} \) is taken which is the set of unit and reference points: 1(1,0,0,0), 2(0,1,0,0), 3(0,0,1,0), 4(0,0,0,1), 13(1,1,1,1). This set contains five points no four of them are on a plane since A intersects any plane in at most three points. Thus A is a (5,3)-arc.

A is a complete (5,3) – arc since every point of PG(3,2) not in A is on a 3-secant; that is, there are no points of index zero for A. This is equivalent to \( C_0 = 0 \).

1.2 The Construction of Complete \( (k,4) \) – arcs in PG(3,2):

The distinct (k,4) – arcs can be constructed by adding to A in each time one point from the remaining ten points of PG(3,2) as follows:

\( A_1 = A \cup \{5\}, A_2 = A \cup \{6\}, A_3 = A \cup \{7\}, A_4 = A \cup \{8\}, A_5 = A \cup \{9\}, A_6 = A \cup \{10\}, A_7 = A \cup \{11\}, A_8 = A \cup \{12\}, A_9 = A \cup \{14\}, A_{10} = A \cup \{15\} \).

By definition 4 of projectively equivalent \( (k,n) \) – arcs, there is only one (6,4) – arc since the arcs \( A_1, \ldots, A_{10} \) are projectively equivalent.

For \( T_0 = 0, T_1 = 2, T_2 = 3, T_3 = 6, T_4 = 4 \). Thus we have \( B = A \cup \{5\} = \{1,2,3,4,5,13\} \) is a complete (6,4) – arc, since every point not in B is on a 4 – secant and B intersects any plane in at most 4 points, that is \( C_0 = 0 \).
1.3 The Construction of Complete \((k,5)\) – arcs in \(PG(3,2)\):

The arc \(B\) is a complete \((6,4)\) – arc. The distinct \((k,5)\) – arcs can be constructed by adding to \(B\) in each time one of the remaining nine points as follows:

\[ B_1 = B \cup \{6\}, \quad B_2 = B \cup \{7\}, \quad B_3 = B \cup \{8\}, \quad B_4 = B \cup \{9\}, \quad B_5 = B \cup \{10\}, \quad B_6 = B \cup \{11\}, \quad B_7 = B \cup \{12\}, \quad B_8 = B \cup \{14\}, \quad B_9 = B \cup \{15\}. \]

By definition 4, there are only two projectively distinct \((7,5)\) – arcs since the arcs \(B_1, B_4, B_5, B_7, B_8, B_9\) are projectively equivalent, for \(T_0 = 0, T_1 = 1, T_2 = 2, T_3 = 5, T_4 = 6, T_5 = 1\) and the arcs \(B_2, B_3, B_6\) are projectively equivalent, for \(T_0 = 0, T_1 = 0, T_2 = 4, T_3 = 5, T_4 = 4, T_5 = 2\). Thus we have two projectively distinct \((7,5)\) – arcs \(C = B \cup \{6\} = \{1,2,3,4,5,6,13\}, \quad D = B \cup \{7\} = \{1,2,3,4,5,7,13\}\).

We try to show the completeness of these arcs. Each of \(C\) and \(D\) is not complete since there exist some points of index zero.

We take the union of \(C\) and \(D\). Then \(E = C \cup D = \{1,2,3,4,5,6,7,13\}\), \(E\) is incomplete \((8,5)\) – arc since there exists one point of index zero for \(E\), which is the point \(15\).

We add the point \(15\) to \(E\), we obtain a complete \((9,5)\) – arc \(F\), \(F = E \cup \{15\} = \{1,\ldots,9,11,13,15\}\). Thus every point not in \(F\) is on a \(5\) – secant and \(F\) intersects any plane at most \(5\) points.

1.4 The Construction of Complete \((k,6)\) – arcs in \(PG(3,2)\):

The arc \(F = \{1,\ldots,7,13,15\}\) is a complete \((9,5)\) – arc. The distinct \((k,6)\) – arcs can be constructed by adding to \(F\) in each time one of the remaining six points, then:

\[ F_1 = F \cup \{8\}, \quad F_2 = F \cup \{9\}, \quad F_3 = F \cup \{10\}, \quad F_4 = F \cup \{11\}, \quad F_5 = F \cup \{12\}, \quad F_6 = F \cup \{14\}. \]

By the definition 4, there are only two projectively distinct arcs since the arcs \(F_1, F_2, F_5, F_6\) are projectively equivalent, for \(T_0 = T_1 = T_2 = 0, T_3 = 2, T_4 = 4, T_5 = 6, T_6 = 3\) and the arcs \(F_3, F_4\) are projectively equivalent, for \(T_0 = T_1 = 2, T_2 = 4, T_4 = 4, T_5 = 7, T_6 = 2\). Thus we have two projectively distinct \((10,6)\) – arcs \(G_1 = \{1,2,3,4,5,6,7,8,13,15\}, \quad G_2 = \{1,2,3,4,5,6,7,11,13,15\}\). Each of them is incomplete since there exist some points of index zero. We take the union of \(G_1\) and \(G_2\), \(G = G_1 \cup G_2 = \{1,2,3,4,5,6,7,8,11,13,15\}\), \(G\) is incomplete \((11,6)\) – arc since there exists one point of index zero, which is the point \(9\), then \(H = G \cup \{9\} = \{1,\ldots,9,11,13,15\}\).

\(H\) is a complete \((12,6)\) – arc, since every point not in \(H\) is on a \(6\) – secant and \(H\) intersects any plane at most \(6\) points.

1.5 The Construction of Complete \((k,7)\) – arcs in \(PG(3,2)\):

The arc \(H = \{1,\ldots,9,11,13,15\}\) is a complete \((12,6)\) – arc. Adding all the remaining points to \(H\), The complete \((15,7)\) – arc can be obtained which is the maximal arc since it contains all points of \(PG(3,2)\), (see figure (1)).

2- The Reverse Construction of Complete \((k,n)\)-Arcs in \(PG(3,2)\):

Complete \((k,n)\) – arcs in \(PG(3,2)\) can be constructed by eliminating some points from the complete arcs of degree \(m\), where \(m = n + 1\), \(3 \leq n \leq 6\), through the following steps:

2.1 The complete \((k,7)\) – arc in \(PG(3,2)\):

The projective space \(PG\) \((3,2)\) contains \(15\) points and \(15\) planes, each plane contains exactly \(7\) points, then the maximal complete \((k,7)\) – arc \(A\) exists when \(k = 15\). This arc contains all the points of \(PG(3,2)\) since it intersects every plane in exactly \(7\) points and hence there arc no points of index zero for \(A\). So \(A = \{1, \ldots, 15\}\) is the complete \((15,7)\) – arc.

2.2 The Construction of Complete \((k,6)\) – arc in \(PG(3,2)\):

A complete \((k,6)\) – arc \(B\) is constructed from the complete \((15,7)\) – arc \(A\) by eliminating some points from \(A\) such that:

1. \(B\) intersects any plane in at most \(6\) points.
2. every point not in B is on at least one 6 – secant of B.

The points 1, 2, 5 are eliminated from A, we obtain a complete (12,6) – arc B, since there are no points of index zero for B. B = \{3, 4, 6, ..., 15\}.

2.3 The Construction of Complete (k,5) – arc in PG(3,2) :

A complete (k,5) – arc in PG (3,2) can be constructed from the complete (12,6) – arc B by eliminating some points from B, which are: 3, 6, 9.

Then a complete (9,5) – arc C is obtained, C = \{4, 7, 8, 10, 11, 12, 13, 14, 15\} since each point not in C is on at least one 5 – secant, hence there are no points of index zero for C and C intersects any plane of PG(3,2) in at most 5 points.

2.4 The Construction of Complete (k,4) – arc in PG(3,2) :

A complete (k,4) – arc in PG(3,2) can be constructed from the complete (9,5) – arc C by eliminating three points from C, which are the points 4, 7, 10, then a complete (6,4) – arc D is obtained, D = \{8, 11, 12, 13, 14, 15\} since each point not in D is on at least one 4 – secant of D and hence there are no points of index zero and D intersects each plane in at most 4 points.

2.5 The Construction of Complete (k,3) – arc in PG(3,2) :

A complete (k,3) – arc in PG(3,2) can be constructed from the complete (6,4) – arc D by eliminating one point from D, which is the point : 15.

A complete (5,3) – arc E is obtained, E = \{8, 11, 12, 13, 14\} since each point not in E is on at least one 3 – secant, hence there are no points of index zero for E and E intersects each plane in at most 3 points.

See figure (2).

3- Results and Conclusion

From the previous results of the two methods, we found that there is no differences between them, the numbers of the points of the complete (k,n) – arcs in the two methods given in table (2).

References

Table (1): The Points $P_i$ and Planes $\pi_i$ of $PG(3,2)$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$P_i$</th>
<th>$\pi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,0,0,0)</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>(0,1,0,0)</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>(0,0,1,0)</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>(0,0,0,1)</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>(1,1,0,0)</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>(0,1,1,0)</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>(1,0,1,1)</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>(1,1,1,1)</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>(1,0,1,0)</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>(1,1,1,0)</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>(0,1,1,1)</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>(1,1,1,1)</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>(1,0,0,1)</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>(1,0,0,1)</td>
<td>15</td>
</tr>
</tbody>
</table>

Table (2): The Maximum $(k,n)$-arcs in Two Methods

<table>
<thead>
<tr>
<th>$n$</th>
<th>maximum $(k,n)$-arcs in the first method</th>
<th>maximum $(k,n)$-arcs in the second method</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>
Fig. (1): All complete \((k_n, n)\) – arcs in PG(3, 2), \(3 \leq n \leq 7\)
Fig. (2): All complete $(k_n, n)$ – arcs in $PG(3, 2)$, $3 \leq n \leq 7$, by reverse construction
البناء والبناء العكسي للأقواس الكاملة للفضاء الثلاثي الاسقاطي حول حقل كالواGF(2)

آمال شهاب المختار
قسم الرياضيات، كلية التربية، ابن الهيثم، جامعة بغداد
استلم البحث في: 11 أيار 2011 قبل البحث في: 16 حزيران 2011

الخلاصة
الهدف الأساسي من هذا البحث هو إيجاد الأقواس الكاملة في الفضاء الثلاثي الأسقاطي حول حقل كالواGF(2) والذي يرمز له (3.2)، بطرقتين ومن ثم نقارن بين الطرقتين.