THE MAXIMUM NUMBER IN WHICH EVERY STRONG TOURNAMENT CONTAINS A TRANSITIVE SUBTOURNAMENTS

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ABSTRACT:
In this paper, we find the maximum number in which every strong tournament contains a transitive subtournaments.

1. Introduction:
Tournaments provide a model of the statistical technique, called the method of paired comparisons. This method is applied when there are a number of objects to be judged on the basis of some criterion and it is impracticable to consider them all simultaneously. The objects are compared two at a time and one member of each pair is chosen. This method and related topics are discussed in K.B Reid [4]. Tournament have also been studied in connecting with sociometric relations in small groups. A survey of some of these investigations is given by R.Fraisse [2]. Our main object here to derive the maximum number where every strong tournament contains a transitive subtournaments.

2. Definitions:
2.1 A tournament $T_n$ consists of $n$ nodes $p_1$, $p_2$, ..........., $p_n$ such that each pair of distinct nodes $p_i$ and $p_j$ is joined by one and only one of the oriented arcs $p_i \rightarrow p_j$ or $p_j \rightarrow p_i$. The relation of dominance thus defined is a complete, irreflexive, antisymmetric, binary relation, every restriction of a tournament is subtournament. [2]

2.2. The score of $p_i$ is the number $s_i$ of nodes that $p_i$ dominates the score vector of $T_n$ in the ordered $n$-tuple $(s_1, s_2, ..........., s_n)$. We usually assume that the nodes are labeled in such a way that $s_1 \leq s_2 \leq ....... \leq s_n$ [2]

2.3 Strong tournament: For any subset $X$ of the nodes of a tournament $T_n$, let
$\Gamma(x) = \{ q : p \rightarrow q \ for some \ p \in X \}$
A tournament $T_n$ is strong if and only if for every node $p$ of $T_n$ the set:
$\{ p \} \cup \Gamma(p) \cup \Gamma^2(p) \cup ......... \cup \Gamma^{n-1}(p)$ contains every node of $T_n$. [2]

2.4 A tournament is transitive if, whenever $p \rightarrow q$ and $q \rightarrow r$, then $p \rightarrow r$. [2]

2.5 A tournament $T_n$ is reducible if it is possible to partition its nodes into two nonempty sets $B$ and $A$ in such a way that all the nodes in $B$ dominate all the nodes in $A$; the tournament is irreducible if this is not possible. [2]

3. NOTATIONS:
1) Let $S(n,k)$ denote the minimum number of strong subtournament $T_k$ that a strong tournament $T_n$ can have. [2]

2) Let $v = v(n)$ is the Largest integer which every strong tournament contains a transitive Subtournmanent. [2]

4. Theorems:
The following theorem gives some properties of a transitive tournaments whose scores $(s_1, s_2, ......., s_n)$ are in nondecreasing order.
Theorem 4.1 [5] The following statements are equivalent.
(1) $T_n$ is transitive.
(2) Node $p_j$ dominates node $p_i$ if and only if $j > i$.
(3) $T_n$ has score $(0, 1, \ldots, n-1)$.
(4) The score vector of $T_n$ satisfies the equation:
$$
\sum_{j=1}^{n} S_j^2 = \frac{n(n-1)(2n-1)}{6}
$$
(5) $T_n$ contains no cycles.
(6) $T_n$ contains exactly $\binom{n}{k+1}$ paths of length $K$, if $1 \leq k \leq n-1$.
(7) $T_n$ contains exactly $\binom{n}{k}$ transitive subtournaments $T_k$, if $1 \leq k \leq n$.
(8) Each principal submatrix of the dominance matrix $M(T_n)$ contains a row and column of zeros.

Every tournament $T_n$ $(n \geq 4)$ contains at least one transitive subtournament $T_3$, but not every tournament $T_n$ is itself transitive.

THEOREM 4.2 [1] if $3 \leq k \leq n$, then
$$
S(n, k) = n - k + 1
$$
THEOREM 4.3 [3] Each node of an irreducible tournament $T_n$ is contained in some $k$-cycle, for $k=3, 4, \ldots, n$.

The main results

THEOREM 4.4 if $\lceil \log_2 n \rceil$ denotes the greatest integer not exceeding $\log_2 n$ then
$$
\lceil \log_2 n \rceil + 1 \leq v(n) \leq \lceil 2 \log_2 n \rceil + 1
$$

Proof: Consider a tournament $T_n$ in which the node $P_n$ has the largest score $S_n$. It must be that $S_n \geq \lceil \frac{1}{2} n \rceil$, so there certainly exists a subtournament $T(\lfloor \frac{1}{2} n \rfloor)$ in $T_n$ each node of which is dominated by $P_n$. We may suppose that $\lceil \log_2 \lceil \frac{1}{2} n \rceil \rceil + 1$ nodes. These nodes together with $P_n$ determine a transitive subtournament of $T_n$ with at least
$$
\lceil \log_2 \lceil \frac{1}{2} n \rceil \rceil + 2 = \lceil \log_2 n \rceil + 1
$$

nodes. The lower bound now follows by induction there are $2^{\binom{n}{2} - \binom{v}{n}}$ tournaments $T_n$, containing a given transitive subtournament $T_v$, and there are $\binom{n}{v}$ such subtournaments $T_v$ possible. Therefore,
$$
\binom{n}{v}!2^{\binom{n}{2} - \binom{v}{n}} \geq 2^{\binom{n}{2}}
$$

Since every tournament $T_n$ contains at least one, transitive subtournament $T_v$, this inequality implies that $n^v \geq 2^{\binom{n}{2}}$. Consequently, $v \leq \lceil 2 \log_2 n \rceil + 1$, and the theorem is proved.

The exact value of $v(n)$ is known only for some small values of $n$. For example, $v(7) \geq 3$. By theorem 4.4 the tournament $T_7$, in which $P_i \rightarrow P_j$ if and only if $j \rightarrow i$ is a quadratic residue modulo 7 contains no transitive subtournament $T_4$. It follows that $v(7) = 3$. We examine other similarly constructed tournaments and we deduced the information about $v(n)$ given in the following table.
Table \( v(n) \), the largest integer \( v \) such that every tournament \( T_n \) contains a transitive subtournament \( T_v \).

\[
\begin{align*}
v(2) &= v(3) = 2 \\
v(4) &= \ldots = v(7) = 3 \\
v(8) &= \ldots = v(11) = 4 \\
4 &\leq v(12) \leq \ldots \leq v(15) \leq 5 \\
v(16) &= \ldots = v(23) = 5 \\
5 &\leq v(24) \leq \ldots \leq v(31) \leq 7 \\
6 &\leq v(32) \leq \ldots \leq v(43) \leq 7
\end{align*}
\]

Let \( u(n,k) \) denote the maximum number of transitive subtournaments \( T_k \) that a strong tournament \( T_n \) can have. (The problem is trivial if \( T_n \) is not strong)

Theorem 4.5 if \( 3 \leq k \leq n \) then \( u(n,k) = \binom{n}{k} - \binom{n-2}{k-2} \)

Proof:- When \( k = 3 \) the theorem follows from theorem 4.2, since every subtournament \( T_3 \) is either strong or transitive. We now show that \( u(n,k) \leq \binom{n}{k} - \binom{n-2}{k-2} \), for any larger fixed value of \( k \). This inequality certainly holds when \( n = k \), if \( n > k \geq 4 \), then any strong tournament \( T_n \), contains a strong subtournament \( T_{n-1} \) by theorem 4.3. Let \( p \) be the node not in \( T_{n-1} \), there are at most \( u(n-1,k) \) transitive subtournaments \( T_k \) of \( T_n \) that contain the node \( p \) and at most \( u(n-1,k-1) \) that do not. We may suppose

\[
u(n-1,k-1) \leq \binom{n-1}{k-1} - \binom{n-3}{k-3},
\]

and

\[
u(n-1,k) \leq \binom{n-1}{k} - \binom{n-3}{k-2}.
\]

Therefore

\[
u(n,k) \leq u(n-1,k-1) + u(n-1,k)
\]

\[
\leq \binom{n-1}{k-1} + \binom{n-1}{k} - \binom{n-3}{k-3} - \binom{n-3}{k-2}
\]

\[
= \binom{n}{k} - \binom{n-2}{k-2}
\]

The inequality now follows by induction:

To show that \( u(n,k) \leq \binom{n}{k} - \binom{n-2}{k-2} \) consider the strong tournament \( T_n \) in which \( p_1 \to p_n \) but otherwise \( p_j \to p_i \) if \( j > i \) (this tournament is illustrated in the following figure) this tournament has exactly \( \binom{n}{k} - \binom{n-2}{k-2} \) transitive subtournament \( T_k \) if \( 3 \leq k \leq n \) because every subtournament \( T_k \) is transitive except those containing both \( p_1 \) and \( p_n \), this completes the proof of the theorem.
Corollary 4.6 the maximum number of transitive subtournaments a strong tournament $T_n$ ($n \geq 3$) can contain, including the trivial tournaments $T_1$ and $T_2$, is $3 \cdot 2^{n-2}$.

Let $r(n, k)$ denote the minimum number of transitive subtournaments $T_k$ a tournament $T_n$ can have. It follows from Theorem 4.4 that $r(n, k) = 0$ if $k > \lceil \log_2 n \rceil + 1$ and that $r(n, k) > 0$ if $k \leq \lfloor \log_2 n \rfloor + 1$.

Theorem 4.7 let

$$r(n, k) = \begin{cases} n \frac{(n-1)(n-3)}{2} \frac{(n-2^{k-1}+1)}{2^{k-1}} & \text{if } n > 2^{k-1} - 1, k \text{ denotes the number of nodes in a subtournament } T_k \\ 0 & \text{if } n \leq 2^{k-1} - 1 \end{cases}$$

Then

$$r(n, k) \geq \tau(n, k)$$

Proof: when $k = 1$, the result is certainly true if we count the trivial tournament $T_1$ as transitive, if $k \geq 2$, then clearly

$$r(n, k) \geq \sum_{i=1}^{n} r(s_i, k-1),$$

where $(s_1, s_2, \ldots, s_n)$ denote the score vector of the tournament $T_n$. We may suppose that $r(s_i, k-1) \geq \tau(s_i, k-1)$; since $\tau(n, k)$ is convex function of $n$ for fixed values of $k$, we may apply Jensen’s inequality and conclude that:

$$r(n, k) \geq \sum_{i=1}^{n} \tau(s_i, k-1) \geq nr(\frac{1}{2}(n-1), k-1) = \tau(n, k).$$

The theorem now follows by induction on $k$.

Notice that the lower bound in Theorem 4.4 follows from Theorem 4.7.

References: