On Artin Cokernel of Dihedral Group $D_n$ When n is an even Number

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Abstract:
In this paper, we find the general form of Artin characters table $\text{Ar}(D_n)$, when $n$ is an even number and the cyclic decomposition of Artin Cokernel $\text{AC}(D_n)$, when $n$ is an even number such that:

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots \cdot p_m^{\alpha_m} \cdot 2^\beta,$$

where $p_i$ are distinct primes for all $i = 1, 2, 3, \ldots, m$.

$$\text{AC}(D_n) = \bigoplus_{i=1}^{m} \bigoplus_{\beta=1}^{C_2}.$$

Introduction:

The abelian group of all $Z$-valued characters of a finite group $G$ under the operation of pointwise addition over the group of induced unit characters form all cyclic subgroups of the group $G$ (Artin characters), $\text{R}(G)/\text{T}(G)$ form a finite abelian group which is called Artin Cokernel of the group $G$. The problem of determining the cyclic decomposition of $\text{AC}(G)$ seem to be untouched. In this work, $G$ is considered to be the dihedral group $D_n$ when $n$ is an even number. To do this work we must do the following steps:

1- we must know the rational valued characters table of the group $D_n$, $\equiv^* (D_n)$.
2- we must find Artin characters table of the group $D_n$, $\text{Ar}(D_n)$.
3- we must find the matrix which expresses the Artin characters of the group $D_n$ in terms of rational valued characters, $M(D_n)$.
4- From (3) we must find the invariant factors matrix $M(D_n)$.
5- From (4) we can find the cyclic decomposition of $\text{AC}(D_n)$.

In 2000 H.R. Yassien [6] studied the cyclic decomposition of $\text{AC}(G)$ when $G$ is an elementary abelian group. In 2002 H.H. Abbass [5] found $\text{Ar}(D_n)$. In 2006 A.S. Abed [2] found $\text{Ar}(C_n)$ when $C_n$ is the cyclic group of order $n$. In this paper, we find $\text{Ar}(D_n)$ and we study $\text{AC}(D_n)$ of the non-abelian group $D_n$ when $n$ is an even number.

1. Some Basic Concepts:
   In this section, we shall give basic concepts, notations and theorems about matrix representation, characters and Artin characters, which will be used in the next sections.

**Definition (1.1):**

The general Linear group $\text{GL}(n,F)$ is a multiplicative group of all non-singular $n \times n$ matrices over the field $F$.

**Example (1.2):**

Consider the field of complex numbers $\mathbb{C}$, $\text{GL}(2,\mathbb{C}) = \{ A: A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; a,b,c,d \in \mathbb{C} \text{ and } A \text{ is a non-singular} \}$

**Definition (1.3):**

A matrix representation of a group $G$ is a homomorphism of $G$ into $\text{GL}(n,F)$, $n$ is called the degree of matrix representation $T$. In particular, $T$ is called a unit representation (principal) if $T(g) = 1$, for all $g \in G$.

**Example (1.4):**

Consider the symmetric group $S_3$, define $T: S_3 \rightarrow \text{GL}(2,\mathbb{C})$ as follows:

$T((1)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T((1\ 2)) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $T((1\ 3)) = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$, $T((2\ 3)) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $T((1\ 2\ 3)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, $T((1\ 3\ 2)) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$. $T$ is a matrix representation of $S_3$ of degree 2.

**Definition (1.5):**

The trace of an $n \times n$ matrix $A$ is the sum of the main diagonal elements, denoted by $\text{tr}(A)$.

**Example (1.6):**

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\text{tr}(A) = 1 + 1 = 2$.

**Definition (1.7):**

Let $T$ be a matrix representation of degree $n$ of a finite group $G$ over the field $F$. The character $\chi$ of degree $n$ of $T$ is the mapping $\chi : G \rightarrow F$ defined by $\chi(g) = \text{tr}(T(g))$ for all $g \in G$. In particular, the character of the principal representation ($\chi(g) = 1$, for all $g \in G$) is called the principal character.

**Example (1.8):**

The character $\chi$ of the matrix representation $T$ in example (1.4) is of degree 2 and it is defined as follows:

$\chi((1)) = 1 + 1 = 2$, $\chi((1\ 2)) = 0 + 0 = 0$, $\chi((1\ 3)) = -1 + 1 = 0$, $\chi((2\ 3)) = 1 + 1 = 0$, $\chi((1\ 2\ 3)) = -1$ and $\chi((1\ 3\ 2)) = -1$.

**Definition (1.9):**

Two elements $g$ and $h$ in $G$ are said to be conjugate if $h = xg x^{-1}$, for some $x \in G$. 

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the relation of conjugacy is an equivalence relation on G. The equivalence classes determined by this relation are referred to as the conjugate classes and $\text{CL}_g$, $g \in G$ is the conjugate class of the element $g$.

Example (1.10):

The two elements $(1\ 2\ 3)$ and $(1\ 3\ 2)$ are conjugate in the symmetric group $S_3$ because $(1\ 2) \cdot (1\ 2\ 3) \cdot (1\ 2)^{-1} = (1\ 3\ 2)$.

Definition (1.11):[3]

The centralizer of $x$ in $G$ is the subgroup $C_G(x) = \{a \in G : a \cdot x \cdot a^{-1} = x\}$.

Example (1.12):

The centralizer of $(1\ 2\ 3)$ in $S_3$ is the subgroup $C_{S_3}((1\ 2\ 3)) = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$.

Definition (1.13):[3]

Let $H$ be a subgroup of $G$ and $\phi$ be a character of $H$, the induced character on $G$ is given by

$$\phi^G(h) = \frac{1}{|H|} \sum_{x \in G} \phi^G(xg^{-1})$$

where $g \in G$ and $\phi^G$ is defined by

$$\phi^G(h) = \begin{cases} 
\phi(h) & \text{if } h \in H \\
0 & \text{if } h \notin H
\end{cases}$$

Example (1.14):

Consider the subgroup $H = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$ of the symmetric group $S_3$, let $\phi$ be the principal character of $H$, i.e $\phi(g) = 1$, for all $g \in G$. We calculate $\phi^{S_3}$ as follows:

$$\phi^{S_3}((1)) = \frac{1}{3} \sum_{x \in S_3} \phi^G(x(1)x^{-1}) = \frac{1}{3} \sum_{x \in S_3} \phi(1) = \frac{1}{3} \cdot 6 = 2.$$  

$$\phi^{S_3}((12)) = \frac{1}{3} \sum_{x \in S_3} \phi^G(x(12)x^{-1}) = \frac{1}{3} \left[ \phi^G((1)(12)(1)) + \phi^G((12)(12)(12)) + \phi^G((12)(13)(12)) + \phi^G((23)(12)(23)) + \phi^G((123)(12)(132)) + \phi^G((132)(12)(123)) \right] = \frac{1}{3} \left[ \phi^G(12) + \phi^G(12) + \phi^G(23) + \phi^G(12) + \phi^G(13) + \phi^G(23) + \phi^G(13) \right] = \frac{1}{3} \cdot 0 = 0.$$  

The following theorem is used to find the induced characters of the cyclic subgroups.

Theorem (1.15):[6]

Let $H$ be a cyclic subgroup of $G$ and $h_1, h_2, \ldots, h_m$ are chosen representatives for the $m$-conjugate classes of $H$ contained in $\text{CL}_g$, $g \in G$, then
\[
\phi^G(g) = \frac{C_G(g)}{C_H(g)} \sum_{i=1}^{m} \phi(h_i) \quad \text{if} \quad h_i \in H \cap \text{CL}_g
\]
\[
\phi^G(g) = 0 \quad \text{if} \quad H \cap \text{CL}_g = \Phi
\]

For the proof, see [6]

**Definition (1.16):**[6]

Let G be a finite group, any character induced from the principal character of cyclic subgroup of G is called Artin character of G.

**Example (1.17):**

The character in example (1.14) is Artin character of the symmetric group $$S_3$$.

**Definition (1.18):**[9]

Two elements of the group G are said to be $$\Gamma$$-conjugate if the cyclic subgroups they generate are conjugate in G, this defines an equivalence relation on G. Its classes are called $$\Gamma$$-classes.

**Example (1.19):**

The two elements (1 2 3) and (1 3 2) are $$\Gamma$$-conjugate in the symmetric group $$S_3$$ because 
\[(1 2) \triangleleft (1 2 3) > (1 2)^{-1} = (1 3 2)\].

**Proposition (1.20):**[14]

The number of all distinct Artin characters on a group G is equal to the number of $$\Gamma$$-classes on G.

For the proof, see [14].

**Definition (1.21):**[2]

The information about Artin characters of a finite group G is displayed in a table called Artin characters table of G, denoted by Ar(G) which is $$l \times l$$ matrix whose columns are $$\Gamma$$-classes and rows the values of all Artin characters on G, where l is the number of $$\Gamma$$-classes

**Example (1.22):**

Given the cyclic group $$C_3 = < r >$$ of order 3, the $$\Gamma$$-classes on $$C_3$$ [1] = {1} and [r] = {r, $$r^2$$}. The Artin characters table of $$C_3$$, Ar ($$C_3$$) =

<table>
<thead>
<tr>
<th>\phi_1</th>
<th>\phi_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

, where $$\phi_1$$ and $$\phi_2$$ are the Artin characters of $$C_3$$.

**Definition (1.23):**[3]

A rational valued character $$\theta$$ of G is a character whose values are in the set of integer numbers Z, which is $$\theta(g) \in Z$$, for all $$g \in G$$.

**Example (1.24):**

The principal character on a finite group G is a rational valued character of G.

**Proposition (1.25):**[12]

The number of all distinct rational valued characters of a finite group G equals to the number of
Definition (1.26): [12]

The information about rational valued characters of a finite group $G$ is displayed in a table called the rational valued characters table of $G$, denoted by $\equiv^*(G)$ which is $l \times l$ matrix whose columns are $\Gamma$-classes and rows are the values of all rational valued characters of $G$, where $l$ is the number of $\Gamma$-classes.

Example (1.27):

$$\equiv^*(C_3) = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix},$$

where $\theta_1$ and $\theta_2$ are the rational valued characters of $C_3$.

Theorem [Artin] (1.28): [9]

Every rational valued character of a finite group $G$ can be written as a Linear combination of Artin's characters with coefficient rational numbers.

For the proof, see [9].

2. The Factor Group $AC(G)$:

The definition of the factor group $AC(G)$ was introduced by T.Y Lam [14] in 1967. The applications of $AC(G)$ not only in the mathematics but also in physics and chemistry.

In this section we shall study $AC(G)$, dihedral group $D_n$ and $\equiv^* (D_n)$, when $n$ is an even number.

Definition (2.1): [14]

Let $R(G)$ be the group of $\mathbb{Z}$-valued generalized characters of $G$ under the operation pointwise addition and $T(G)$ is a normal subgroup of $R(G)$ generated by Artin's characters. The abelian factor group $R(G)/T(G)$ is called Artin's Cokernel of $G$, denoted by $AC(G)$.

Definition (2.2): [12]

Let $M$ be a matrix with entries in a principle domain $R$. A $K$-minor of $M$ is the determinant of $K \times K$ submatrix preserving row and column order.

Definition (2.3): [12]

A $K$-th determinant divisor of $M$ is the greatest common divisor (g.c.d) of all $K$-minor, denoted by $D_K(M)$.

Theorem (2.4): [12]

Let $M$ be an $n \times n$ matrix with entries in a principle domain $R$, then there exist matrices $P$ and $W$ such that

1. $P$ and $W$ are invertibles.
2. $P.M.W = D$.
3. $D$ is a diagonal matrix.
4. If we denote $D_{ij}$ by $d_j$ then there exists a natural number $m$; $0 \leq m \leq n$ such that $j > m$ implies $d_j = 0$ and $j < m$ implies $d_j \neq 0$ and $1 < j < m$ implies $d_j/d_{j+1}$.

For the proof, see [12].
Definition (2.5):[12]
Let M be a matrix with entries in a principal domain R, and equivalent to matrix
\[ D = \{ d_1, d_2, ..., d_m, 0, 0, ..., 0 \} \], Such that \( d_j/d_{j+1} \) for \( 1 < j < m \), D is called the invariant factor matrix of M and \( d_1, d_2, ..., d_m \) the invariant factors of M.

Remark (2.6):-
According to the Artin theorem (1.28) there exists an invertible matrix \( M^{-1}(G) \) with entries in the field of rational \( Q \) such that
\[ \equiv^{*} (G) = M^{-1}(G) \cdot Ar(G) \] and this implies
\[ M(G) = Ar(G).(\equiv^* (G))^{-1} \]
By theorem (2.4) there exists two matrices \( P(G) \) and \( W(G) \) such that
\[ P(G) \cdot M(G) \cdot W(G) = \text{diag} \{ d_1, d_2, ..., d_l \} = D(G) \], where \( d_j = \pm D_j(M(G))/D_j-1(M(G)) \) and \( l \) is the number of \( \Gamma \)-classes.

Theorem (2.7):[6]
\[ AC(G) = \bigoplus_{j=1}^{l} \mathbb{C}_{d_j} \] where \( d_j = \pm D_j(M(G))/D_j-1(M(G)) \), and \( l \) is the number of all distinct \( \Gamma \)-classes and \( \mathbb{C}_{d_j} \) is cyclic subgroup of order \( d_j \).
For the proof, see [6].

Proposition (2.8):[12]
Let \( P \) be a prime number, then the rational valued characters table of cyclic group \( C_p^s \) is:
\[ \theta = r \] is given by
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{\( \Gamma \)-Classes} & [1] & [r_{p^{s-1}}] & [r_{p^{s-2}}] & [r_{p^{s-3}}] & \ldots & [r_{p^2}] & [r] \\
\hline
\theta_1 & p^{s-1}(p-1) & -p^{s-1} & 0 & 0 & \ldots & 0 & 0 & 0 \\
\theta_2 & p^{s-2}(p-1) & p^{s-2}(p-1) & -p^{s-2} & 0 & \ldots & 0 & 0 & 0 \\
\theta_3 & p^{s-3}(p-1) & p^{s-3}(p-1) & p^{s-3}(p-1) & -p^{s-3} & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\theta_{s-1} & p(p-1) & p(p-1) & p(p-1) & p(p-1) & \ldots & p(p-1) & -p & 0 \\
\theta_s & p-1 & p-1 & p-1 & p-1 & \ldots & p-1 & p-1 & -1 \\
\theta_{s+1} & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
\hline
\end{array}
\]
\[ \equiv^* (C_p^s) = \]
For the proof, see [12].
Remark (2.9):- In general if \( n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_m^{\alpha_m} \) where \( p_1, p_2, \ldots, p_m \) are distinct primes, then
\[ \equiv^* (C_n) = \equiv^* (C_{p_1}^{\alpha_1}) \otimes \equiv^* (C_{p_2}^{\alpha_2}) \otimes \ldots \otimes \equiv^* (C_{p_m}^{\alpha_m}) \] where \( \otimes \) is the tensor product.
Definition (2.10):[9]

The dihedral group $D_n$ is a certain non-abelian group of order $2n$, it is usually thought as a group of transformations of Euclidean plane of regular $n$-polygon consisting of rotation $r^k$ (about the origin) with angle $2\pi k/n$ and reflections $sr^k$ (a cross lines through the origin).

In general it can be written as $D_n = \{S^i r^k : 0 \leq k \leq n-1, 0 \leq i \leq 1\}$, where $r^n = 1, S^2 = 1, Sr^k = r^{-k}$.

The cyclic group of order $n$, $C_n = \langle r \rangle$ is a normal subgroup of $D_n$.

Proposition (2.11):[5]

The rational valued characters table of $D_n$ when $n$ is an even number is given by:

$$
\begin{array}{c|cccc}
\Gamma \text{-Classes} & \Gamma \text{-Classes of } C_n & [r^0] & [s] & [sr] \\
\hline
0_1 & & & & \\
0_2 & & & & \\
\vdots & & & & \\
0_{l-1} & & & & \\
0_l & & & & \\
0_{l+1} & & & & \\
0_{l+2} & & & & \\
\end{array}
$$

Where $0_{l+2}(r^k) = 1$ if $k$ is an even number.

and $0_{l+2}(r^k) = -1$ if $k$ is an odd number.

$l$ is the number of $\Gamma$-Classes of $C_n$.

For the proof, see [5].

Theorem (2.12):[2]

Let $p$ be a prime number, then the Artin characters table of $C_p^s$ is given by:

$$
\begin{array}{c|cccc}
\Gamma \text{-Classes} & [1] & [r^{p^{s-1}}] & [r^{p^{s-2}}] & \cdots & [r] \\
\hline
\varphi_1 & p^s & 0 & 0 & \cdots & 0 \\
\varphi_2 & p^{s-1} & p^{s-1} & 0 & \cdots & 0 \\
\varphi_3 & p^{s-2} & p^{s-2} & p^{s-2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\varphi_s & p & p & p & \cdots & 0 \\
\varphi_{s+1} & 1 & 1 & 1 & \cdots & 1 \\
\end{array}
$$

For the proof, see [2].

Remark (2.13):

Let $n$ be any positive integer and
\[ n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m} \] where \( p_1, p_2, \ldots, p_m \) are distinct primes, then

\[ \text{Ar}(C_n) = \text{Ar}(C_{p_1}^{\alpha_1}) \otimes \text{Ar}(C_{p_2}^{\alpha_2}) \otimes \cdots \otimes \text{Ar}(C_{p_m}^{\alpha_m}) \]

Where \( \otimes \) is the tensor product.

Proposition (2.14):[13]

If \( P \) be a prime number and \( S \) is a positive integer, then

\[ M(C_{p^i}) = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix} \text{ and } P(C_{p^i}) = \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 1 & -1 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & -1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}. \]

For the proof, see [13]

Remark (2.15):

In general if \( n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m} \cdot 2^\beta \) where \( p_1, p_2, \ldots, p_m \) are distinct primes, then

1. \( P(C_n) = P(C_{p_1}^{\alpha_1}) \otimes P(C_{p_2}^{\alpha_2}) \otimes \cdots \otimes P(C_{p_m}^{\alpha_m}) \otimes P(C_{2^\beta}) \)

We can write

2. \[ M(D_n) = \begin{bmatrix}
\beta \text{ times} & \begin{bmatrix}
1 & 1 \\
1 & 1 \\
\vdots & \vdots \\
1 & 1
\end{bmatrix} \\
R_2(C_n) & \begin{bmatrix}
1 & 1 \\
1 & 1 \\
\vdots & \vdots \\
1 & 1
\end{bmatrix} \\
\beta \text{ times} & \begin{bmatrix}
1 & 1 \\
1 & 1 \\
\vdots & \vdots \\
1 & 1
\end{bmatrix} \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix} \]

Where \( R_2(C_n) \) is the matrix obtained by omitting the last two rows and columns \( \{0,0,\ldots,0,1,1\} \) and \( \{1,1,\ldots,1,0,0,\ldots,0\} \) from the tensor product

\[ M(C_{p_1}^{\alpha_1}) \otimes M(C_{p_2}^{\alpha_2}) \otimes \cdots \otimes M(C_{p_m}^{\alpha_m}) \otimes M(C_{2^\beta}). \]

3. The Main Results:-

This section is devoted to study the Artin characters \( \text{Ar}(D_n) \) and the cyclic decomposition of the factor group \( \text{Ac}(D_n) \), when \( n \) is an even number.

Theorem (3.1):
The Artin characters table of the dihedral group $D_n$ when $n$ is an even number $\text{Ar}(D_n) = \Gamma$-

<table>
<thead>
<tr>
<th>Classes</th>
<th>$\frac{n}{2}$</th>
<th>$\frac{n}{2}$</th>
<th>$\frac{n}{2}$</th>
<th>$\frac{n}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[C_{D_n}(\text{CL}_a)]$</td>
<td>2n</td>
<td>2n</td>
<td>n</td>
<td>n</td>
</tr>
<tr>
<td>$\Phi_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi_2$</td>
<td>2Ar($C_n$)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi_j$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi_{l+1}$</td>
<td>n</td>
<td>0</td>
<td>...........</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi_{l+2}$</td>
<td>n</td>
<td>0</td>
<td>...........</td>
<td>0 2 0</td>
</tr>
</tbody>
</table>

Where $l$ is the number of $\Gamma$-classes of $C_n$.

Proof: Let $g \in D_n$ and $\phi_j$ is the Artin characters of $C_n$ for all $j=1,2,\ldots, l$

Case (I):
If $H$ is a subgroup of $C_n=<r>$, $1 < j < l$ and the principal character of $H$, then applying theorem (1.15) yields

$$\Phi_j(g) = \frac{[C_{D_n}(g)]}{[C_H(g)]} \cdot \sum_{i=1}^{m} \phi(h_i)$$

(i) If $g = 1$

$$\Phi_j(1) = \frac{[C_{D_n}(1)]}{[C_H(1)]} \cdot \phi(1) = \frac{2n}{[C_H(1)]} \cdot 1 = \frac{2[C_{C_n}(1)]}{[C_H(1)]} = 2\phi_j(1)$$ since $H \cap \text{CL}(1) = \{1\}$

(ii) If $g = r^2, g \neq 1$ and $g \in H$

$$\Phi_j(g) = \frac{[C_{D_n}(g)]}{[C_H(g)]} \cdot \phi(1) = \frac{2n}{[C_H(g)]} \cdot 1 \quad \text{since} \quad H \cap \text{CL}(g) = \{g\} \quad \text{and} \quad \phi(g) = 1$$

$$= \frac{2[C_{C_n}(g)]}{[C_H(g)]} \cdot \phi(g) = 2\phi_j(g)$$

(iii) If $g \neq r^2$ and $g \in H$

$$\Phi_j(g) = \frac{[C_{D_n}(g)]}{[C_H(g)]} \left( \phi(g) + \phi(g^{-1}) \right)$$

$$= \frac{n}{[C_H(g)]} (1 + 1) \quad \text{since} \quad H \cap \text{CL}(g) = \{g,g^{-1}\} \quad \text{and} \quad \phi(g) = \phi(g^{-1}) = 1$$

$$= \frac{2[C_{C_n}(g)]}{[C_H(g)]} \cdot \phi(g) = 2\phi_j(g)$$
(iv) If \( g \notin H \)
\[ \Phi_j(g) = 0 \quad \text{since} \quad H \cap \text{CL}(g) = \Phi \]
\[ = 2.0 = 2.\phi'(g). \]

Case (II):
If \( H = \langle S \rangle = \{1,S\} \)
(i) If \( g = 1 \)
\[ \Phi_{i+1}(1) = \frac{C_{D_1}(1)}{C_H(1)} \cdot \phi(1) = \frac{2n}{2} \cdot 1 = n \]
(ii) If \( g = S \)
\[ \Phi_{i+1}(S) = \frac{C_{D_1}(S)}{C_H(S)} \cdot \phi(1) = \frac{2^2}{2} \cdot 1 = 2 \]
Otherwise
\[ \Phi_{i+1}(g) = 0 \quad \text{since} \quad H \cap \text{CL}(g) = \Phi \]

Case (III):
If \( H = \langle S_r \rangle = \{1,S_r\} \)
(i) If \( g = 1 \)
\[ \Phi_{i+1}(1) = \frac{C_{D_1}(1)}{C_H(1)} \cdot \phi(1) = \frac{2n}{2} \cdot 1 = n \quad \text{since} \quad H \cap \text{CL}(1) = \{1\} \]
(ii) If \( g = S_r \)
\[ \Phi_{i+1}(S_r) = \frac{C_{D_1}(S_r)}{C_H(S_r)} \cdot \phi(1) = \frac{2^2}{2} \cdot 1 = 2 \quad \text{since} \quad H \cap \text{CL}(S_r) = \{S_r\} \]
Otherwise
\[ \Phi_{i+1}(g) = 0 \quad \text{since} \quad H \cap \text{CL}(g) = \Phi \]

Theorem (3.2):
If \( n \) is an even number and \( n = \prod_{i=1}^{\alpha_1} \cdot \prod_{i=1}^{\alpha_2} \cdot \cdots \cdot \prod_{i=1}^{\alpha_m} \cdot 2^\beta \) where
\( P_1, P_2, \ldots, P_m \) are distinct primes and \( P_i \neq 2 \) for all \( i=1,2,\ldots,m \), then the cyclic decomposition of \( AC(D_n) \) is
\[ (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1)(\beta+1) - 1 \]
\[ AC(D_n) = \bigoplus_{i=1}^{C_2} \]

Proof:-
By theorem (3.1) we obtained the Artin's characters table \( Ar(D_n) \) and from proposition (2.11) we can find the rational valued characters table \( \equiv^*(D_n) \).

Thus, by the definition of the matrix \( M(D_n) \) (Remark (2.6))
We have \( M(D_n) = \text{Ar}(D_n).(\equiv^*(D_n))^{-1} \), then
By using theorem (2.7)

Then

By theorem (2.4) and remark (2.6) we can take

square matrix.

Which is \([\alpha_1 + 1] \cdot (\alpha_2 + 1) \cdots (\alpha_n + 1)(\beta + 1) + 2\] \times \([\alpha_1 + 1] \cdot (\alpha_2 + 1) \cdots (\alpha_n + 1)(\beta + 1) + 2\]

By theorem (2.4) and remark (2.6) we can take

and

Where \( k = [(\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdots (\alpha_n + 1)(\beta + 1) - 1] \) and \( I_k \) is the identity matrix of order \( k \times k \).

Then

\( P(D_n) \cdot M(D_n) \cdot W(D_n) = D(D_n) = \text{diag} \{2,2,2,\ldots,2,1,1,1\} \)

\( = \{d_1, d_2, \ldots, d_{(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_n+1)(\beta+1)-1}, 1, 1, 1\} \)

By using theorem (2.7)
\[
(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1)(\beta+1)-1 \quad (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1)(\beta+1)-1
\]

\[AC(D_n)=\bigoplus_{i=1}^{C_{d_i}}=\bigoplus_{i=1}^{C_2}
\]

\[C_2\]
since \(d_i=2\) for all \(i=1,2,\ldots, [(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1)(\beta+1)-1]
\]

Example (3.3):
To find \(AC(D_{1800})\) , \(AC(D_{365904})\)

\[AC(D_{1800})=AC(D_{3^2\times 5\times 7})=\bigoplus_{i=1}^{C_2}=\bigoplus_{i=1}^{C_2}
\]

\[C_2=119\]

\[AC(D_{365904})=AC(D_{1^1\times 3^7\times 5^2})=\bigoplus_{i=1}^{C_2}=\bigoplus_{i=1}^{C_2}
\]

References :-