Solving Euler's Equation by Using New Transformation

By
Assis.prof.Ali Hassan Mohammed
Kufa University. College of Education . Department of Computer Sciences
And
Assis.Lect.Athera Nema kathem
Kufa University. College of Education. Department of Mathematics

Abstract
In this paper, we introduce definition of new transformation which we call it Temimi transformation .Also, we introduce properties ,theorems, proofs and transformations of the polynomials functions ,logarithms functions and other functions .Also ,we introduce how we can use this transformation and it's inverse to solve the Euler's equation [2 ] .

المستخلص
في هذا البحث قدمنا تعريف لتحويل جديد أسمينا تحويل التميمي كلاً ذلك قدمنا خصائص ،نظريات ،براهين وتحويلات لمتعددات الحدود والدوال اللوغاريتمية ودوال أخرى لأجل استعمال هذا التحويل ومعوكسه في حل معادلة أولير [ 2 ].

Introduction
Laplace transformation[1]is considered as one of the important transformations which is known to solve the L.O.D.E.with constants coefficients and which has the general formula

\[ a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = f(x) \]

With one condition that Laplace transformation of the function f(x) is defined .In this paper, we define a new transformation which is work to solve the L.O.D.E with variables coefficients( Euler's equation) which has the general form

\[ a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = f(x) \]

This transformation is defined for some functions for example constant functions, logarithm functions , polynomial functions and other functions .

Definition 1
The Al-Temimi transformation for the function f(x) (x >1) is defined by the following integral :

\[ T(f(x)) = \int_{1}^{\infty} x^{-p} f(x) dx = F(p) \]

such that this integral is convergent , P is constant .

Property of this transformation
This transformation is characterized by the linear property ,that is

\[ T[Af(x) + Bg(x)] = AT[f(x)] + BT[g(x)] \]

where A,B are constants ,the functions f(x),g(x) are defined when x>1 .
Proof:

\[ T[Af(x) + Bg(x)] = \int_1^\infty (Af(x) + Bg(x))x^{-p}dx = A\int_1^\infty x^{-p}f(x)dx + B\int_1^\infty x^{-p}g(x)dx = AT\{f(x)\} + BT\{g(x)\} \]

Transformations for some functions

We are going to find the Temimi transformation for some functions, like the fixed functions, logarithm functions, polynomial functions and other functions.

1. If \( f(x) = 1 \), then

\[ T\{1\} = \frac{1}{p-1} \]

2. If \( f(x) = k \), \( x > 1 \) then

\[ T\{k\} = \frac{k}{p-1} \]

3. If \( f(x) = x^n \), \( n \in \mathbb{R} \) then

\[ T\{x^n\} = \int_1^\infty x^{-p}x^n\ dx = \int_1^\infty x^{-p+n}\ dx = \frac{x^{-p+n+1}}{-p+n+1}|_1^\infty = \frac{1}{p-(n+1)} \]

4. If \( f(x) = \ln x \) then

\[ T\{\ln x\} = \int_1^\infty x^{-p}\ln x\ dx = \int_1^\infty \frac{x^{-p+1}\ln x}{-p+1}|_1^\infty - \int_1^\infty \frac{x^{-p+1}}{x-\ln x}\ dx \]

\[ = -\frac{1}{-p+1}\int_1^\infty \frac{x^{-p+1}}{(-p+1)(-p+1)}\ dx = \frac{1}{(p-1)^2} \]

5. If \( f(x) = x^n \ln x \), \( n \in \mathbb{R} \) then

\[ T\{x^n\ln x\} = \int_1^\infty x^{-p}x^n\ln x\ dx = \int_1^\infty x^{-p+n}\ln x\ dx = \int_1^\infty \frac{x^{-p+n+1}}{-p+n+1}\ln x|_1^\infty - \int_1^\infty \frac{x^{-p+n}}{-p+n+1}\ dx \]

\[ = \frac{-1}{-p+n+1}\int_1^\infty \frac{x^{-p+n+1}}{-p+n+1}|_1^\infty - \frac{1}{p-(n+1)}\int_1^\infty \frac{1}{-p+n+1}\ dx \]

\[ = \frac{-1}{-p+n+1}\left[ \frac{x^{-p+n+1}}{-p+n+1}\right]_1^\infty - \frac{1}{p-(n+1)} \]

\[ = \frac{1}{(p-1)^2+a^2} \]

6. If \( f(x) = \sin(a \ln x) \) then

\[ T\{\sin(a \ln x)\} = \int_1^\infty \frac{a}{(p-1)^2+a^2} \]

\[ \sin(a \ln x)\ dx \]

\[ = \left[ \frac{x^{-p+1}}{-p+1}\sin(a \ln x)\right]_1^\infty - \int_1^\infty \frac{a}{-p+1} x^{-p}\cos(a \ln x)\ dx \]

\[ \{T\{\sin(a \ln x)\}\} = \int_1^\infty x^{-p}\sin(a \ln x)\ dx \]

\[ \quad \text{proof} \]

\[ \quad \text{proof} \]

\[ \quad 104 \]
\[(1 + \frac{a^2}{(p-1)^2}) \int_1^\infty x^{-p} \sin(a \ln x)dx = \frac{a}{(p-1)^2}\]
\[= \frac{a}{(p-1)^2} - \frac{a^2}{(p-1)^2} \int_1^\infty x^{-p} \sin(a \ln x)dx \Rightarrow \]
\[\therefore \int_1^\infty x^{-p} \sin(a \ln x)dx = \frac{a}{(p-1)^2 + a^2}\]
\[; p>1 \text{, and } a \text{ be constant } T\{\cos(a \ln x)\} = \frac{(p-1)}{(p-1)^2 + a^2} \]

7- \text{If } f(x) = \cos (a \ln x) \text{ then}
\[= \frac{-1}{p+1} \int_1^\infty x^{-p} \sin(a \ln x)dx \]
\[= \frac{-1}{p+1} a \int_1^\infty x^{-p} \sin(a \ln x)dx \]
\[= \int_1^\infty x^{-p} \cos(a \ln x)dx \]
\[\Rightarrow (1 + \frac{a^2}{(p-1)^2}) \int_1^\infty x^{-p} \cos(a \ln x)dx = \frac{1}{(p-1)} \Rightarrow \]
\[; |p-1| > a \quad \frac{a}{(p-1)^2 - a^2} \quad \text{T\{sinh(a \ln x)\}} = \]

8- \text{If } f(x) = \sinh(a \ln x) \text{ then}
\[\int_1^\infty x^{-p} \sinh(a \ln x)dx = \int_1^\infty x^{-p} \left(\frac{e^{a \ln x} - e^{-a \ln x}}{2}\right)dx = \frac{1}{2} \int_1^\infty (x^{-p+a} - x^{-(p+a)})dx \]
\[\text{proof : } \text{T\{sinh(alnx)\}} = \]
\[\frac{1}{2} \left[ \frac{1}{p-a-1} - \frac{1}{p+a-1} \right] = \frac{a}{(p-1)^2 - a^2} \]
\[; |p-1| > a \quad \frac{(p-1)}{(p-1)^2 - a^2} \quad \text{T\{cosh(alnx)\}} = \]

9- \text{If } f(x) = \cosh(a \ln x) \text{ then}
\[\int_1^\infty x^{-p} \cosh(a \ln x)dx = \int_1^\infty x^{-p} \left(\frac{e^{a \ln x} + e^{-a \ln x}}{2}\right)dx \]
\[\text{T\{cosh(alnx)\}} = \]
\[\frac{1}{2} \int_1^\infty (x^{-p+a} + x^{-(p-a)})dx = \frac{(p-1)}{(p-1)^2 - a^2} \]

\text{Theorem (1)}
\[T\{x^{-a} f(x)\} = F(p + a) \text{If } T\{f(x)\} = F(p) \text{ and } a \text{ is constant ,then}
\]
\[T\{x^{-a} f(x)\} = \int_1^\infty x^{-p} x^{-a} f(x)dx = \int_1^\infty x^{-(p+a)} f(x)dx = F(p + a)\]
proof :
Definition 2
Let f(x) be a function where (x>1) and T{f(x)}=F(p) , f(x) is said to be an inverse for the Temimi transformation and written as : T⁻¹{F(p)}= f(x)
where T⁻¹ returns the transformation to the original function .

; p>1  T(1) = \frac{1}{p-1} since \quad T⁻¹{\frac{1}{p-1}} = 1
For example

; p>1 T\{k\} = \frac{k}{p-1} since \quad T⁻¹{\frac{k}{p-1}} = k
Also

; p>(n+1) \quad \frac{1}{p-(n+1)} since \quad T\{x^n\} = \quad T⁻¹{\frac{1}{p-(n+1)}} = x^n
And

\quad T\{\ln x\} = \frac{1}{(p-1)^2} since \quad T⁻¹{\frac{1}{(p-1)^2}} = \ln x
\quad T\{x^a \ln x\} = \frac{1}{(p-(n+1))^2}; P > (n+1) since \quad T⁻¹{\frac{1}{(p-(n+1))^2}} = x^a \ln x
\quad T\{\sin(a \ln x)\} = \frac{a}{(p-1)^2 + a^2}; p > 1 since \quad T⁻¹{\frac{a}{(p-1)^2 + a^2}} = \sin(a \ln x)
\quad T\{\cos(a \ln x)\} = \frac{(p-1)}{(p-1)^2 + a^2} since \quad T⁻¹{\frac{(p-1)}{(p-1)^2 + a^2}} = \cos(a \ln x)
\quad T\{\sinh(a \ln x)\} = \frac{a}{(p-1)^2 - a^2} since \quad T⁻¹{\frac{a}{(p-1)^2 - a^2}} = \sinh(a \ln x)
\quad T\{\cosh(a \ln x)\} = \frac{(p-1)}{(p-1)^2 - a^2} since \quad T⁻¹{\frac{(p-1)}{(p-1)^2 - a^2}} = \cosh(a \ln x)

A property of T⁻¹ is the linear property as it is for the transformation T .

If T⁻¹{F_1(p)}=f_1(x),...,T⁻¹{F_n(p)}=f_n(x) and a_1,...,a_n are constants then ,
T⁻¹\{a_1F_1(p)+a_2F_2(p)+...+a_nF_n(p)\}=a_1f_1(x)+a_2f_2(x)+...+a_nf_n(x)

T⁻¹{F(a + p)} = x^a T⁻¹{F(p)} \quad \text{If } T⁻¹{F(p)}=f(x) \quad \text{then} \quad \textbf{Theorem (2)}

Proof :
T⁻¹{F(a + p)} = x^a \cdot f(x) = x^a T⁻¹{F(p)}

For example to find the inverse of the terms, we get

\quad T⁻¹{\frac{1}{p+1}} = \frac{1}{x^2 + 1}
\[ T^{-1}\left\{ \frac{1}{(p-2)^2} \right\} = x \ln x - 2 \]

\[ = T^{-1}\left\{ \frac{(p-1) + 2}{(p-1)^2 + 4} \right\} = T^{-1}\left\{ \frac{p-1}{(p-1)^2 + 4} \right\} + T^{-1}\left\{ \frac{2}{(p-1)^2 + 4} \right\} = T^{-1}\left\{ \frac{p+1}{(p^2 - 2p + 5)} \right\} = \cos(2\ln x) + \sin(2\ln x) \]

**Solving the Linear Differential Equations with Variable Coefficients**

One of the most important applications of the Temimi transformation is solving the linear differential equations with variable coefficients. Transforming the L.O.D.E. into algebraic equation in \( y(p) \) is done by transforming the derivations and their coefficients and the function \( f(x) \) to the new formulas. After transforming the \( x^n \frac{d^n y}{dx^n}, \ldots, x \frac{dy}{dx} \) equation into an algebraic equation, we are going to find the inverse transformation for this algebraic equation and the result will be the solution of the differential equation.

\[ a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_0 y = f(x) \]

**Definition 3:**

The equation where \( a_0, \ldots, a_n \) are constants and \( f(x) \) is a function of \( x \), is called Euler's equation.

**Theorem (3):**

If the function \( f(x) \) is defined for \( x > 1 \) and its derivatives \( f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n)}(x) \) are exist then \( T \{ x^n f^{(n)}(x) \} = -f^{(1)}(1) - (p-n)f^{(n-2)}(1) - \ldots - (p-n)(p-n-1) \ldots (p-2)f(1) + (p-n)!F(p) \)

\[ = x^{-p+1} f^{(1)}(x) \bigg|_{1}^{\infty} - \int_{1}^{\infty} (-p + 1) x^{-p} f(x) dx; \quad p > 1 \]

\[ T \{ x f^{(1)}(x) \} = \int_{1}^{\infty} x^{-p+1} f^{(1)}(x) dx \quad \text{proof:} \]

\[ = -f^{(1)}(1) + (p-1)F(p) \]

\[ = x^{-p+2} f^{(1)}(x) \bigg|_{1}^{\infty} - (-p + 2) \int_{1}^{\infty} x^{-p+1} f^{(1)}(x) dx \quad T \{ x^2 f^{(2)}(x) \} = \int_{1}^{\infty} x^{-p+2} f^{(2)}(x) dx \]

\[ = f^{(1)}(1) - (p-2)f(1) + (p-2)(p-1)!F(p) \]

\[ = f^{(1)}(1) - (p-3)f^{(1)}(1) - (p-3)(p-2)f(1) + (p-3)(p-2)(p-1)!F(p) \]

\[ T \{ x^3 f^{(3)}(x) \} = \int_{1}^{\infty} x^{-p+3} f^{(3)}(x) dx \]

Also, thus by repeating this method for \( n \)-times, we get

\[ T \{ x^n f(x) \} = -f^{(n)}(1) - (p-n)f^{(n-2)}(1) - (p-n)(p-n-1)f^{(n-4)}(1) \ldots - (p-n)(p-n-1)(p-n-2) \ldots (p-2)f(1) + (p-n)!F(p) \]

**Example (1):**

To find the solution of the differential equation \( y(1) = y'(1) = 0 \):

\[ x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = x \log x \]

we take \( T \) to both sides of this equation, we get

107
\[
\frac{1}{(p - 2)^2} - f^{(1)}(1) - (p-2)f(1) + (p-2)(p-1)F(p) + 4(p-1)F(p)-4f(1)+2F(p) =
\]
\[
\ldots(1)
\]
\[
\frac{1}{p(p + 1)(p - 2)^2} F(p) = \frac{1}{(p - 2)^2} \Rightarrow F(p)[p^2 + p] =
\]
By using partial fractions for the right side of the equation (1), we get
\[
\ldots(2) \frac{1}{p(p + 1)(p - 2)^2} = \frac{A}{(p-2)} + \frac{B}{(p-2)^2} + \frac{C}{(p+1)} + \frac{D}{p}
\]
\[
\therefore A + B + D = 0 \ldots(3)
\]
\[
-A + B - 4C - 3D = 0 \ldots(4)
\]
\[
2A + B + 4C = 0 \ldots(5)
\]
\[
4D = 1 \ldots(6)
\]
By solving these four equations for getting the constants values, we find that
\[
A = \frac{-5}{36}, \quad B = \frac{1}{6}, \quad C = \frac{-1}{9}, \quad D = \frac{1}{4}.
\]
We substitute this values in equation (2), we get
\[
F(p) = \frac{-5}{36} + \frac{1}{6} + \frac{-1}{9} + \frac{1}{p}
\]
Now, we can find the solution of the original equation by taking \(T^{-1}\) to both sides of the above equation, we get
\[
y = -\frac{5}{36}x + \frac{1}{6}x \ln x - \frac{1}{9x^2} + \frac{1}{4x}
\]

**Example (2):**
To find the solution of the differential equation
\[
y(1) = 2, \quad y'(1) = -4 \quad x^2 y'' + 6xy + 6y = \frac{1}{x^2}
\]
we take \(T\) to both sides of this equation and we get
\[
\frac{1}{(p + 1)} - f^{(1)}(1) - (p-2)f(1)+(p-1)(p-2)F(p)+6(p-1)F(p)-6f(1)+6F(p) =
\]
\[
\frac{2p^2 + 6p + 5}{(p + 1)^2} F(p)(p^2 + 3p + 2) = \frac{1}{(p + 1)} \Rightarrow F(p)(p^2 + 3p + 2) - 2p - 4 =
\]
\[
F(p) = \frac{2p^2 + 6p + 5}{(p + 1)^2(p + 2)}
\]
By using partial fractions for the right side of the above equation, we get
\[
\frac{2p^2 + 6p + 5}{(p + 1)^2(p + 2)} = \frac{A}{(p + 1)} + \frac{B}{(p + 1)^2} + \frac{C}{(p + 2)}
\]
\[
A + C = 2 \ldots(7)
\]
\[
3A + B + 2C = 6 \ldots(8)
\]
From the above equations, we get
\[
A = B = C = 1
\]
\[
F(p) = \frac{1}{(p + 1)} + \frac{1}{(p + 1)^2} + \frac{1}{(p + 2)}
\]
\[
108
\]
And by taking $T^{-1}$ to both sides of the above equation, we get

$$T^{-1}\{F(p)\} = T^{-1}\\left\{\frac{1}{(p+1)} + \frac{1}{(p+1)^2} + \frac{1}{(p+2)}\right\}$$

$$\therefore y = \frac{1}{x^2} + \frac{\ln x}{x^2} + \frac{1}{x^3}$$

So

**Example (3):**

To solve the differential equation

$$xy' + y = 16 \sin (\ln x) \quad y(1) = -7$$

we take $T$ to both sides of the above equation, we get

$$-f(1) + (p-1) F(p) + F(p) = \frac{16}{(p-1)^2 + 1} \quad \Rightarrow \quad pF(p) = \frac{1}{(p-1)^2 + 1} - 7 = \frac{-7p^2 + 14p + 2}{p^2 - 2p + 2}$$

$$\Rightarrow F(p) = \frac{-7p^2 + 14p + 2}{p(p^2 - 2p + 2)}$$

By using partial fractions for the right side of the above equation, we get

$$F(p) = \frac{A}{p} + \frac{Cp + D}{p^2 - 2p + 2} \quad \ldots (10)$$

$$A + C = -7 \quad \ldots (11)$$

$$-2A + D = 14 \quad \ldots (12)$$

$$2A = 2 \quad \ldots (13)$$

After solving these equations, we find $A = 1, C = -8, D = 16$.

We substitute these values in equation (10) we get

$$F(p) = \frac{1}{p} + \frac{8}{(p-1)^2 + 1} - \frac{8(p-1)}{(p-1)^2 + 1}$$

And by taking the inverse transformation of the last equation, we get the solution of the required differential equation

$$y = \frac{1}{x} + 8 \sin(\ln x) - 8 \cos(\ln x)$$

**References**
