المقاسات الجزئية الأولية المضادة

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الخلاصة

في هذا البحث درسنا المقاسات الجزئية الأولية المضادة واعطينا العديد من الخواص المتعلقة بهذا المفهوم.

الكلمات المفتاحية: المقاسات الجزئية الأولية المضادة - المقاسات الجزئية الثانوية المضادة (المضادة الأولية) - المقاسات الثانية.
Coprime Submodules

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Abstract
Let R be a commutative ring with unity and let M be a unitary R-module. Let N be a proper submodule of M, N is called a coprime submodule if \( \frac{M}{N} \) is a coprime R-module, where \( \frac{M}{N} \) is a coprime R-module if for any \( r \in R \), either \( \frac{rM}{N} = 0 \) or \( \frac{M}{N} = \frac{M}{N} \).

In this paper we study coprime submodules and give many properties related with this concept.

Key words: Coprime submodules, second submodule, second (coprime) module, secondary module.

Introduction
Let R be a commutative ring with unity and let M be a unitary R-module. It is well-known that a proper submodule N of an R-module M is called prime if whenever \( r \in R, x \in M, rx \in N \) implies \( x \in N \) or \( r \in [N:M] \), where \([N:M] = \{ r \in R : rM \subseteq N \} \). M is called a prime module if \( \text{ann}_R M = \text{ann}_R N \) for all nonzero submodule N of M, equivalently M is a prime module iff (0) is a prime submodule.

S. Yassem in [7], introduced the notions of second submodules and second modules, where a submodule N of M is called second if for any \( r \in R \), the homothety \( r^* \in \text{End} M \), is either zero or surjective, where \( r^*(m) = rm, \forall m \in M \). It follows that N is a second submodule iff for each \( r \in R \), either \( rN = 0 \) or \( rN = N \). M is called a second module if M is a second submodule of itself.

For an R-module M, the following statements are equivalent:

1. M is a second module.
2. For each \( r \in R \), either \( rM = 0 \) or \( rM = M \).
3. \( \text{ann}_R M = \text{ann}_R N \) for all proper submodules N of M.
4. \( \text{ann}_R M = \text{ann}_R N \) for all fully invariant submodules N of M.
5. modules N of M.
6. \( \text{ann}_R M = W(M), \) where \( W(M) = \{ r \in R : r^* \in \text{End} M, r^* \text{ is not surjective} \} \).

Notice (1) \( \Leftrightarrow \) (2) is clear, (1) \( \Leftrightarrow \) (5) [7, lemma 1.2], (1) \( \Leftrightarrow \) (3) [3, theorem 2.1.6], (3) \( \Leftrightarrow \) (4) [6, theorem 1.3.2].

Notice that statement (3) and statement (4) are used to define coprime module by S. Annin in [2] and I.E Wijayart in [6], respectively.
Moreover Rasha in [3] studied coprime modules and give some generalizations of these modules, (see [3]).

J.Abuhilail in [1], introduced the notion of coprime submodule, where a proper submodule N of M is called coprime if \( \text{ann } \frac{M}{N} = W(\frac{M}{N}) \); that is N is a coprime submodule if \( \frac{M}{N} \) is a coprime R-module.

Our aim in this paper is to study coprime submodules, we give the basic properties about this concept. Also, we study coprime submodules in certain classes of modules.

1- Coprime Submodules

We give the basic properties related with coprime submodules. Also, we study their behaviour in certain classes of modules.

Following J.Abuhilail in [1], a proper submodule N of an R-module M is called coprime if \( \frac{M}{N} \) is a coprime R-module.

An ideal I of a ring R is called coprime ideal iff \( \frac{R}{I} \) is a coprime R-module.

1.1 Remarks and Examples:

(1) N is coprime submodule iff for each \( r \in R \) either \( \frac{rM}{N} = O_M = N \) or \( \frac{rM}{N} = \frac{M}{N} \), that is N is a coprime submodule if for each \( r \in R \), either \( r \in [N:M] \) or for any \( m \in M \), there exists \( m' \in M \) such that \( m - rm' \in N \).

(2) Z is a coprime submodule of the Z-module Q, since \( \frac{Q}{Z} \) is a coprime Z-module [4], [6]. Note that Z is not coprime Z-module, since when \( r = 2 \neq 0 \), \( 2Z \neq Z \).

(3) Every submodule N of the Z-module \( Z_{p^2} \) is a coprime submodule, since \( Z_{p^2}/N \cong Z_{p^2} \) and \( Z_{p^2} \) is a coprime Z-module, hence \( Z_{p^2}/N \) is a coprime Z-module.

(4) Let M be a coprime R-module, then every proper submodule N of M is a coprime submodule.

proof: Since M is a coprime R-module, then by [3,cor. 2.1.12], \( \frac{M}{N} \) is a coprime R-module, for all \( N \ neq M \). Hence N is a coprime submodule.

(5) If N is a maximal submodule of an R-module M, then N is a coprime submodule.

proof: Since N is maximal, \( \frac{M}{N} \) is a simple R-module, hence \( \frac{M}{N} \) is a coprime R-module. Thus N is a coprime submodule.

(6) The converse of (4) is not true in general for example, Z is a coprime submodule of the Z-module Q (see 1.1 (2)) but Z is not a maximal submodule of Q.

(7) Let M be an R-module, let I be an ideal of R such that \( I \subseteq \text{ann } M \), let \( N \neq M \). Then \( N \) is a coprime R-submodule of \( M \iff N \) is a coprime \( \bar{R} \)-submodule of M, where \( \bar{R} = R / I \).
proof: ($\Rightarrow$) Let $N$ be a coprime $R$-submodule. Then $\frac{M}{N}$ is a coprime $R$-module and hence by [3, cor. 2.1.9], $\frac{M}{N}$ is coprime $\overline{R}$-module. Thus $N$ is a coprime $\overline{R}$-module.

($\Leftarrow$) The proof is similarly.

1.2 Proposition:
If $N$ is a coprime submodule, then $[N:M]$ is a prime ideal.

proof: Since $N$ is a coprime submodule, $\frac{M}{N}$ is coprime $R$-module. Hence $\text{ann} \frac{M}{N}$ is a prime ideal of $R$ [3, note 2.1]. But $\text{ann} \frac{M}{N} = [N:M]$, so $[N:M]$ is a prime ideal.

Recall that an $R$-module $M$ is called secondary if for each $r \in R$, either $rm = 0$ or $r^n M = M$, for some $n \in \mathbb{Z}_+$. [7].

We have the following:

1.3 Proposition:
Let $M$ be a secondary $R$-module, let $N < M$. Then $N$ is a coprime submodule iff $[N:M]$ is a prime ideal of $R$.

proof: ($\Rightarrow$) It follows by prop. 1.2.

($\Leftarrow$) Since $M$ is a secondary $R$-module, then $\frac{M}{N}$ is a secondary $R$-module. But $[N:M] = \text{ann} \frac{M}{N}$ is a prime ideal, so by [3,prop.1.2.6], $\frac{M}{N}$ is a coprime $R$-module, hence $N$ is a coprime submodule.

1.4 Proposition:
Let $N$ be a proper submodule of an $R$-module $M$. Then $N$ is a coprime submodule iff $[N:M] = [W:M]$ for all $W \supseteq N$.

proof: If $N$ is a coprime submodule, then $\frac{M}{N}$ is a coprime $R$-module. Hence $\text{ann} \frac{M}{N} = \text{ann} \frac{M}{W}$ for all $W \supseteq N$. It follows that $\text{ann} \frac{M}{N} = \text{ann} \frac{M}{W}$; that is $[N:M] = [W:M]$.

If $[N:M] = [W:M]$, for all $W \supseteq N$, then $\text{ann} \frac{M}{N} = \text{ann} \frac{M}{W}$. But $\frac{M}{W} \cong \frac{M}{N}$, so $\text{ann} \frac{M}{N} = \text{ann} \frac{M}{W}$ and $\frac{M}{N}$ is a coprime $R$-module. Thus $N$ is a coprime submodule.

1.5 Proposition:
Let $W$ be a coprime submodule of $M$ and let $N < M$ such that $N \supset W$. Then $N$ is a coprime submodule of $M$ and $\frac{N}{W}$ is a coprime submodule of $\frac{M}{W}$.
proof: Since $W$ is a coprime submodule, then $M/W$ is a coprime R-module. Hence by [Rem and Ex. 1.1 (4)], $N/W$ is a coprime submodule of $M/W$. Also $M/W$ is a coprime R-module implies $(M/W) / (N/W)$ is a coprime R-module [3,cor. 2.1.12]. But $(M/W) / (N/W) \cong M / N$, hence $M/N$ is a coprime module by [3, Cor. 2.1.14]. Thus $N$ is a coprime submodule of $M$.

1.6 Proposition:

Let $M$ be an R-module, let $N$, $W$ be proper submodules of $M$, $N \subseteq W$ such that $N/W$ is a coprime submodule of $M/W$. Then $N$ is a coprime submodule of $M$.

proof: Since $N/W$ is a coprime submodule of $M/W$, we have $(M/W) / (N/W)$ is a coprime module. Thus $M/N$ is a coprime module and so $N$ is a coprime submodule of $M$.

The following results follow directly by proposition 1.5.

1.7 Corollary:

If $N$ is a coprime submodule of an R-module $M$, $I$ an ideal of R. Then $[N : I]$ is a coprime submodule of $M$.

1.8 Corollary:

Let $A$, $B$ be proper submodules of an R-module $M$. If $A$ or $B$ is a coprime submodule and $A + B \neq M$. Then $A + B$ is a coprime submodule of $M$.

1.9 Proposition:

Let $I$ be a proper ideal of a ring R. Then $I$ is a coprime ideal iff $I$ is a maximal ideal of R.

proof: If $I$ is a coprime ideal of R, then $R/I$ is a coprime R-module. But $R/I$ is a multiplication R-module, so by [3,Rem. And Ex. 2.1.3(5)] $R/I$ is simple R-module. Thus $I$ is a maximal ideal of R.

The converse follows by (Rem. And Ex. 1.1.(5)).

1.10 Corollary:

Let $R$ be a ring. The following are equivalent:

(1) $(0)$ is a coprime submodule of $R$.
(2) $R/(0) \subseteq R$ is a coprime ring (that is $R$ is a field).
(3) $(0)$ is a maximal ideal of $R$.

1.11 Corollary:

Let $R$ be a PID, let $I$ be a nonzero proper ideal of $R$. Then the following are equivalent:

(1) $I$ is a coprime ideal of $R$.
(2) $I$ is a maximal ideal of $R$.
(3) $I$ is a prime ideal of $R$.

1.12 Note:

If $N$ is a coprime submodule of an R-module $M$. Then it is not necessary that $[N:M]$ is a coprime ideal of $R$, as the following example shows:

$Z$ is a coprime submodule of the $Z$-module $Q$ but $[Z:Q] = (0)$ is not a maximal ideal of $Z$, that is $(0)$ is not coprime ideal of $Z$.

1.13 Proposition:

Let $M$ be a multiplication R-module, let $N$ be a proper submodule of $M$. Then $N$ is a coprime submodule iff $[N:M]$ is a coprime ideal of $R$. 
proof: If $N$ is a coprime submodule of $M$, then $\frac{M}{N}$ is a coprime $R$-module. But $M$ is a multiplication $R$-module implies $\frac{M}{N}$ is a multiplication $R$-module. Hence by [3, Rem. and Ex. 2.1.3(5)] $\frac{M}{N}$ is a simple $R$-module. Thus $N$ is a maximal submodule of $M$ which implies that $[N:M]$ is a maximal ideal. Then by prop. 1.9, $[N:M]$ is a coprime ideal.

Conversely, if $[N:M]$ is a coprime ideal of $R$, then by prop. 1.9, $[N:M]$ is a maximal ideal of $R$. Now $M$ is a multiplication module and $[N:M]$ is a maximal ideal of $R$ implies that $N=[N:M]M$ is a maximal submodule of $M$. Thus by Rem. and Ex. 1.1 (5), $N$ is a coprime submodule of $M$.

1.14 Corollary:

Let $M$ be a multiplication $R$-module and let $N < M$. The following are equivalent:

1. $N$ is a coprime submodule of $M$.
2. $[N:M]$ is a coprime ideal of $R$.
3. $[N:M]$ is a maximal ideal of $R$.
4. $N$ is a maximal submodule of $M$.

proof: (1) $\iff$ (2) it follows by prop. 1.13.
(2) $\iff$ (3) it follows by prop. 1.9.
(4) $\implies$ (1) by Rem. and Ex. 1.1 (5).
(3) $\implies$ (4) Since $M$ is multiplication, and $[N:M]$ is a maximal ideal, then $N$ is a maximal submodule of $M$.

The following result shows that a homomorphic image of a coprime submodule is a coprime submodule.

1.15 Theorem:

Let $\psi: M \longrightarrow M'$ be an $R$-epimorphism, let $N < M$. If $N$ is a coprime submodule of $M$, then $\psi(N)$ is a coprime submodule of $M'$.

proof: To prove $\psi(N)$ is a coprime submodule of $M'$, we must prove $\frac{M'}{\psi(N)}$ is a coprime $R$-module, so we must show that $r \frac{M'}{\psi(N)} = \frac{M'}{\psi(N)}$ for all $r \notin \text{ann} \frac{M'}{\psi(N)}$. First $r \notin \text{ann} \frac{M'}{\psi(N)}$, means that $r \notin [\psi(N):M']$. It is easy to check that $[N:M] \subseteq [\psi(N):M']$. Hence $r \notin [N:M] = \text{ann} \frac{M}{N}$. On the other hand $N$ is a coprime submodule, implies $\frac{M}{N}$ is a coprime $R$-module. Hence $\frac{M}{N} = \frac{M}{N}$ since $r \notin \text{ann} \frac{M}{N} = [N:M]$. Now, let $y + \psi(N) \in \frac{M'}{\psi(N)}$, so $y = \psi(m)$ for some $m \in N$, since $\psi$ is an epimorphism. Thus $y + \psi(N) = \psi(m) + \psi(N) = \psi(m + N)$. Hence there exists $m' \in M$ such that $m + N = r m' + N$, so $y + \psi(N) = \psi(r m' + N) = r \psi(m') + N = r (\psi(m') + N) \in r \frac{M'}{\psi(N)}$. Thus $r \frac{M'}{\psi(N)} = \frac{M'}{\psi(N)}$ and so $\frac{M'}{\psi(N)}$ is a coprime $R$-module. Hence $\psi(N)$ is a coprime submodule of $M'$.

Now, we turn our attention to direct sum of coprime submodules.
1.16 Theorem:
Let $M_1$, $M_2$ be $R$-modules, let $N_1 < M_1$, $N_2 < M_2$ such that $\text{ann} \frac{M_1}{N_1} = \text{ann} \frac{M_2}{N_2}$. Then $N_1 \oplus N_2$ is a coprime submodule of $M$ iff $N_1$ is a coprime submodule of $M_1$, $N_2$ is a coprime submodule of $M_2$.

**proof:** ($\Rightarrow$) Let $p_1: M_1 \oplus M_2 \longrightarrow M_1$, $p_2: M_1 \oplus M_2 \longrightarrow M_2$ be the natural projection. Hence $p_1(N_1 \oplus N_2) = N_1$, $p_2(N_1 \oplus N_2) = N_2$ and so by theorem 1.15, $N_1$ is a coprime submodule of $M_1$, $N_2$ is a coprime submodule of $M_2$.

Conversely, to prove $N_1 \oplus N_2$ is a coprime submodule of $M_1 \oplus M_2$. Since $N_1$, $N_2$ are coprime submodules of $M_1$, $M_2$ respectively, then $\frac{M_1}{N_1}$ and $\frac{M_2}{N_2}$ are coprime $R$-module and since $\text{ann} \frac{M_1}{N_1} = \text{ann} \frac{M_2}{N_2}$ it follows that $\frac{M_1 \oplus M_2}{N_1 \oplus N_2}$ is a coprime $R$-module (see [7], [3,prop. 2.3.3]). But it is easy to check that $\frac{M_1 \oplus M_2}{N_1 \oplus N_2} \cong \frac{M_1}{N_1} \oplus \frac{M_2}{N_2}$. Hence by [3,cor. 2.1.14], $\frac{M_1 \oplus M_2}{N_1 \oplus N_2}$ is a coprime $R$-module. Thus $N_1 \oplus N_2$ is a coprime submodule of $M_1 \oplus M_2$.

1.17 Remark:
The condition $\text{ann} \frac{M_1}{N_1} = \text{ann} \frac{M_2}{N_2}$ is necessary condition in Th. 14, as the following example shows:

Consider the $Z$-module $Z$. Let $N_1 = 2Z$, $N_2 = 3Z$, $N_1$, $N_2$ are maximal submodules of $Z$, so $N_1$, $N_2$ are coprime submodules of $Z$ (see Rem. 1.1(5)). Let $N = N_1 \oplus N_2 = 2Z \oplus 3Z < Z \oplus Z$. It is clear that $\text{ann} \frac{Z}{N_1} \neq \text{ann} \frac{Z}{N_2}$. Now $\frac{Z \oplus Z}{N_1 \oplus N_2} \cong \frac{Z}{N_1} \oplus \frac{Z}{N_2} \oplus \frac{Z_2 \oplus Z_3 \oplus Z_6}{N_1 \oplus N_2}$.

But $Z_6$ is not a coprime $Z$-module, so $\frac{Z \oplus Z}{N_1 \oplus N_2}$ is not a coprime $Z$-module. Thus $N_1 \oplus N_2$ is not a coprime submodule of $Z \oplus Z$.

The following property explains the behaviour of coprime submodules under localization.

1.18 Proposition:
Let $S$ be a multiplicative subset of a ring $R$. Let $N$ be a proper submodule of an $R$-module $M$ such that $S^{-1}N \neq S^{-1}M$. If $N$ is a coprime submodule of $M$, then $S^{-1}N$ is coprime submodule of $S^{-1}M$.

**proof:** $N$ is a coprime submodule of $M$ implies $\frac{M}{N}$ is a coprime $R$-module, then by [3,prop.2.1.38], $S^{-1}\left(\frac{M}{N}\right)$ is a coprime $S^{-1}R$-module. But [5,lemma 9.12,p.173],
Recall that an R-module $M$ is antihopfian if $M = M/N$ for all $N < M$ (4). Hence we get the following result directly.

**1.19 Remark:**
Let $M$ be an antihopfian R-module. Then every submodule of $M$ is coprime submodule.

**proof:** Since $M \cong \frac{M}{N}$, $\text{ann } M = \text{ann } \frac{M}{N}$, that is $M$ is coprime R-module. Then by (Rem. and Ex. 1.1(4)) every proper submodule is coprime submodule.

**1.20 Proposition:**
Let $M$ be a finitely generated R-module, let $N < M$. If $N$ is a coprime submodule, then $N$ is prime.

**proof:** Since $N$ is a coprime submodule, $M/N$ is a coprime R-module. But $M$ is a finitely generated R-module, so $M/N$ is finitely generated. Hence by [3, Th. 2.4.8], $M/N$ is a prime R-module and hence $O_{MN} = N$ is a prime submodule of $\frac{M}{N}$. It follows that $N$ is a prime submodule of $M$.

**1.21 Remark:**
The condition $M$ is finitely generated in prop. 2.1 is necessary condition, as the following example shows.

$Z$ is a coprime submodule of the $Z$-module $Q$ and $Q$ is not finitely generated. Also $Z$ is not a prime submodule of $Q$.

**1.22 Corollary:**
Let $M$ be a Noetherian coprime R-module, then every proper submodule of $M$ is prime.

**proof:** It follows directly by prop. 1.20.

**1.23 Proposition:**
Let $M$ be an R-module such that $rM \cap N = rN$ for all $r \in R$ and for all $N < M$. Then every prime submodule is a coprime submodule.

**proof:** Let $N$ be a prime submodule of $M$. Let $W \supset N$. We shall prove that:

\[
\frac{M}{N} \cap \frac{W}{N} = \frac{W}{N}
\]

as follows: let $x \in \frac{M}{N} \cap \frac{W}{N}$, so $x = w + N = r(m + N)$ for some $w \in W$, $m \in M$. Hence $r m - w \in N \subseteq W$. Thus $r m \in W$, which implies that $r m \in rM \cap W = rW$ and hence $r m = r y$ for some $y \in W$. Then $r m + N = r y + N$, that is $r(m + N) = r(y + N) \in \frac{W}{N}$. Thus

\[
\frac{M}{N} \cap \frac{W}{N} = \frac{W}{N}
\]

On the other hand, $N$ is a prime submodule of $M$ implies $\frac{M}{N}$ is a prime R-module. Then by [3, prop. 2.4.1, p.54] $\frac{M}{N}$ is a coprime R-module and hence $N$ is a coprime submodule.

**1.24 Corollary:**
Let $R$ be a regular ring (in sense of Von Neumann), let $M$ be an R-module. Then every prime submodule of $M$ is a coprime submodule of $M$. 

proof: Since $R$ is a regular ring, implies $rM \cap N = rN$ for all $r \in R$ and for all $N < M$, then the result is obtained by prop.1.23.

References