Chained fuzzy modules

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Abstract  
Let R be a commutative ring with unity. In this paper we introduce the notion of chained fuzzy modules as a generalization of chained modules. We investigate several characterizations and properties of this concept.

Introduction  
In this paper we introduce the concept of chained fuzzy modules as a generalization of the concept (chained modules) in ordinary algebra. This paper consists of three sections:  
In section one, we recall some basic definitions and results which we needed later.  
In section two, we give some results about chained fuzzy modules such as its relationship with its levels.  
Section three is devoted for studying the direct sum of chained fuzzy modules.  
Finally, we study the homomorphic image and inverse of chained fuzzy modules.

1. Preliminaries  
The following definitions and results are needed later.

1.1 Definition, [1]  
Let M be a nonempty set and I be the closed interval [0,1] of the real line (numbers). A fuzzy set A in M (a fuzzy subset A of M) is a function from M into I.

1.2 Definition, [2]  
Let x: M → [0,1] be a fuzzy set in M, where x ∈ M, t ∈ [0,1] defined by:  
\[ X_t(y) = \begin{cases}  
  1 & \text{if } x = y \\
  0 & \text{if } x \neq y 
\end{cases} \]

for all y ∈ M, x_t is called a fuzzy singleton or fuzzy point in M, if x=0 and t=1, then  
\[ 0_t(y) = \begin{cases}  
  1 & \text{if } y = 0 \\
  0 & \text{if } y \neq 0 
\end{cases} \]

We shall call such fuzzy singleton the fuzzy zero singleton.

1.3 Definition, [2]  
Let A and B be two fuzzy sets in M, then  
1. A = B if and only if A(x) = B(x), for all x ∈ M.  
2. A ⊆ B if and only if A(X) ≤ B(X), for all x ∈ M.  
3. (A ∩ B)(x) = min{A(x), B(x)} for all x ∈ M.  
4. (A ∪ B)(x) = max{A(x), B(x)} for all x ∈ M.

1.4 Definition, [3]  
Let A be a fuzzy set in M and t ∈ [0,1]. The set  
\[ A_t = \{ x ∈ M | A(x) ≥ t \} \]

is called level subset of A.

1.5 Remark (1)  
The following properties of level subsets hold for each t ∈ [0,1]  
1. (A ∩ B)_t = A_t ∩ B_t  
2. A = B if and only if A_t = B_t, for all t ∈ [0,1].
Where $A$ and $B$ are fuzzy sets.

Now, we can give the definition of image and inverse image of a fuzzy set.

1.6 Definition, [4]

Let $f$ be a mapping from a set $M$ into a set $N$, $A$ be a fuzzy set in $M$ and $B$ be a fuzzy set in $N$. The image defined by:

$$f(A) = \begin{cases} \sup \{A(z) \mid z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

where $f^{-1}(y) = \{x : f(x) = y\}$ and the inverse image of $B$, denoted by $f^{-1}(B)$, is the fuzzy set in $M$ defined by:

$$f^{-1}(B)(x) = B(f(x)),$$

for all $x \in M$.

1.7 Definition, [5]

Let $f$ be a mapping from a set $M$ into a set $M'$. A fuzzy subset $A$ of $M$ is called $f$-invariant if $A(x) = A(y)$ whenever $f(x) = f(y)$, where $x, y \in M$.

The following lemma is needed in section three.

1.8 Lemma, [5]

If $f$ is a function defined on a set $M$, $A_1$ and $A_2$ are fuzzy subset of $M$, $B_1$ and $B_2$ are fuzzy subset of $f(M)$. Then the following are true:

1. $A_1 = f^{-1}(f(A_1))$, whenever $A_1$ is $f$-invariant.
2. $f(f^{-1}(B_1)) = B_1$.
3. if $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.
4. if $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

1.9 Definition, [6]

Let $(R, +, \cdot)$ be a ring and let $X$ be a fuzzy set in $R$. Then $X$ is called a fuzzy ring in ring $(R, +, \cdot)$ if and only if, for each $x, y \in R$

1. $X(x+y) \geq \min \{X(x), X(y)\}$
2. $X(x) = X(-x)$
3. $X(xy) \geq \min \{X(x), X(y)\}$.

1.10 Definition, [7]

A fuzzy subset $X$ of a ring $R$ is called a fuzzy ideal of $R$, if for each $x, y \in R$

1. $X(x-y) \geq \min \{X(x), X(y)\}$
2. $X(xy) \geq \max \{X(x), X(y)\}$.

1.11 Definition, [2]

Let $M$ be an $R$-module. A fuzzy set $X$ of $M$ is called a fuzzy module of $M$ if

1. $X(x-y) \geq \min \{X(x), X(y)\}$, for all $x, y \in M$.
2. $X(rx) \geq X(x)$, for all $x \in M$ and $r \in R$.
3. $X(0) = 1$.

1.12 Definition, [6]

Let $X$ and $A$ be two fuzzy modules of an $R$-module $M$. $A$ is called a fuzzy submodule of $X$ if $A \subseteq X$.

1.13 Proposition, [7]

Let $A$ be a fuzzy set of $M$. Then the level subset $A_t$, $t \in (0,1]$ is a submodule of $M$ if and only if $A$ is a fuzzy submodule of $X$ where $X$ is a fuzzy module of $M$ such that $A(x) \leq X(x)$, $\forall x \in M$.

1.14 Definition, [8]

A fuzzy module $X$ of an $R$-module $M$ is called fuzzy simple if and only if $X$ has no fuzzy proper submodules.

1.15 Definition, [8]

A fuzzy module $X$ of an $R$-module $M$ is called fuzzy cyclic module, if there exists $x_0 \subseteq X$ such that each $y \subseteq X$ written as $y = r_1 x_0$, for some fuzzy
singleton $r_t$ of $R$ where $k, \ell, t \in [0,1]$. In this case, we shall write $X=(x_t)$ to denote the fuzzy cyclic module generated by $x_t$.

1.16 Definition [6]
Let $X$ and $Y$ be two fuzzy modules of $R$-modules $M_1$ and $M_2$ respectively, $f:X \rightarrow Y$ is called a fuzzy homomorphism if $f:M_1 \rightarrow M_2$ is $R$-homomorphism and $y(f(x))=X(x)$ for each $x \in M_1$.

1.17 Remark [9]
1- Let $M$ and $M'$ be two $R$-modules, $f:M \rightarrow M'$ be an epimorphism. If $A$ is a fuzzy submodule of $M$, then $f(A)$ is a fuzzy submodule of $M'$.
2- Let $M$ and $M'$ be two $R$-modules, $f:M \rightarrow M'$ be a homomorphism. If $B$ is a fuzzy submodule of $M'$, then $f^{-1}(B)$ is a fuzzy submodule of $M$.

1.18 Definition [2]
Suppose $A$ and $B$ be two fuzzy modules of $R$-module $M$. We define $(A:B)$ by:
$$(A:B)=\{r_t; r_t \text{ is a fuzzy singleton of } R \text{ such that } r_t B \subseteq A \text{ and } (A:B)(t)=\sup \{t \in [0,1] | r_t B \subseteq A\}$$
for all $r \in R$. If $B=(b_k)$, $(A:(b_k))=\{r_t; r_b b_k \subseteq A, r_t \text{ is a fuzzy singleton of } R\}$.

1.19 Definition [10]
Let $X$ and $Y$ be two fuzzy modules of $M_1$, $M_2$ respectively. Define $X \oplus Y:M_1 \oplus M_2 \rightarrow [0,1]$ by
$$(X \oplus Y)(a,b)=\min\{X(a),Y(b)\}$$
for all $(a,b) \in M_1 \oplus M_2$. $X \oplus Y$ is called a fuzzy external direct sum of $X$ and $Y$.

1.20 Proposition [10]
Let $X$ and $Y$ are fuzzy modules of $M_1$ and $M_2$ respectively, then $X \oplus Y$ is a fuzzy module of $M_1 \oplus M_2$.

1.21 Proposition [10]
Let $A$ and $B$ be two fuzzy submodules of a fuzzy module $X$ such that $X=A \oplus B$, then $X_s=A_s \oplus B_s$, for all $s \in (0,1]$.

2. Chained Fuzzy Module

In this section we introduce the concept of chained fuzzy module. Some basic results of this concept are considerate.

2.1 Definition, [11]
An $R$-module $M$ is called chained module if for each submodules $A$, $B$ of $M$, either $A \subseteq B$ or $B \subseteq A$.

We fuzzify this definition as follows:

2.2 Definition
Let $X$ be a fuzzy module of an $R$-module $M$ then $X$ is called a chained fuzzy module if for each fuzzy submodules of $X$ either $A \subseteq B$ or $B \subseteq A$.

To prove our next theorem, first we prove the following lemma:

2.3 Lemma
Let $A$ and $B$ be two fuzzy subset of $R$ then $A \subseteq B$ if and only if $A_t \subseteq B_t$, for each $t \in [0,1]$.

Proof: It is easy so it is omitted.

The following theorem characterizes chained fuzzy module in terms of it is level module.

2.4 Theorem
A fuzzy module $X$ of an $R$-module $M$ is a chained if and only if $X_t$ is a chained module, $\forall t \in (0,1]$.

Proof: If $X$ is chained fuzzy module. To prove $X_t$ is chained module $\forall t \in (0,1]$.
Let $I, J$ be submodules of $X_t$. Define:
$$A(x) = \begin{cases} t & x \in I \\ 0 & x \not\in I \end{cases}, \quad B(x) = \begin{cases} t & x \in J \\ 0 & x \not\in J \end{cases}$$
A, B are fuzzy submodules of X. But A_t=I, B_t=J since X is chained fuzzy module, then either A \subseteq B or B \subseteq A. Hence A_t \subseteq B_t or B_t \subseteq A_t (by lemma (2.3)). Thus I \subseteq J or J \subseteq I.

Conversely, if X_t is chained module, to prove X is a chained fuzzy module, let A, B fuzzy submodules in X. Then A_t, B_t are submodules in X_t, for all t \in [0,1] since X_t is chained R-module then A_t \subseteq B_t or B_t \subseteq A_t which implies A \subseteq B or B \subseteq A (lemma (2.3)).

2.5 Examples

1. Let X(x)=1 for all x \in \mathbb{Z}_8 X_t=\mathbb{Z}_8 for all t \in [0,1]. But \mathbb{Z}_8 is chained. Hence by theorem (2.4) X is a chained fuzzy module.

2. Every fuzzy simple module is a chained fuzzy module.

2.6 Remark

If Y \subseteq X and X is a chained fuzzy module then Y is a chained fuzzy module.

Proof: Let A, B be two fuzzy submodules of Y then A, B are fuzzy submodules of X, since X is chained fuzzy module. Then A \subseteq B or B \subseteq A which implies Y is a chained fuzzy module.

2.7 Definition, [10]

A fuzzy module X is called uniform fuzzy module if A \cap B \neq 0 \quad (\forall t \in (0,1]).

2.8 Proposition

A fuzzy module X of an R-module M is uniform if and only if X_t is a uniform module, \forall t \in (0,1].

Proof: If X is uniform fuzzy module, to prove X_t is uniform module \forall t \in (0,1]. Let I, J be submodules of X_t. Define

A(x) = \begin{cases} \frac{t}{4} & x \in I \\ 0 & x \notin I \end{cases}, \quad B(x) = \begin{cases} \frac{t}{4} & x \in J \\ 0 & x \notin J \end{cases}

A, B are fuzzy submodules of X. But A_t=I, B_t=J since X is uniform fuzzy module, then A \cap B \neq 0. Hence (A \cap B)_t \neq 0_t which implies A_t \cap B_t \neq 0_t (by remark 1.5). This I \cap J \neq 0_t.

Conversely, if X_t is uniform module, to prove X is a uniform fuzzy module, let A, B fuzzy submodules in X. Then A_t, B_t are submodules in X_t, for all t \in (0,1], since X_t is uniform R-module then A_t \cap B_t \neq 0_t which implies (A \cap B)_t \neq 0_t (by remark 1.5). Thus A \cap B \neq 0_t.

Now, we shall show the relationship between uniform fuzzy module and chained fuzzy module as the following proposition:

2.9 Proposition

Every chained fuzzy module is a uniform fuzzy module.

Proof: Let X be a chained fuzzy module of an R-module M then A \subseteq B or B \subseteq A
if A \subseteq B then A \cap B= A
if B \subseteq A then A \cap B=B
which implies A \cap B=0_t.

2.10 Remark

The converse of proposition (2.9) is not true for the following example shows:

2.11 Example

Let M=Z as a Z-module

X:M \rightarrow [0,1] such that X(x)=1 \forall x \in M X_t=z for all t \in [0,1]. But Z is uniform. Hence by proposition (2.9) X is a uniform fuzzy module. But X is not chained fuzzy module since \exists A, B fuzzy submodules of X defined by

A(x) = \begin{cases} \frac{1}{4} & x \in (2) \\ \frac{1}{4} & x \notin (2) \end{cases}, \quad B(x) = \begin{cases} \frac{1}{4} & x \in (5) \\ \frac{1}{4} & x \notin (5) \end{cases} \quad \text{and} \quad A \nsubseteq B \text{ and } B \nsubseteq A.

Recall that if A and B are two submodules of an R-module M, then A and B are called comparable if A \subseteq B and B \subseteq A.

We shall fuzzify this concept as follows:
2.12 Definition
Let A, B be two fuzzy submodules of a fuzzy module X of an R-module M, then A and B are called comparable if $A \subseteq B$ and $B \subseteq A$.

2.13 Proposition
A fuzzy module X of an R-module M is chained iff every two cyclic fuzzy submodules of X are comparable.

**Proof:** Let A and B be fuzzy submodules of X. Suppose $A \nsubseteq B$, we show $B \nsubseteq A$ since $A \subseteq B$, there exists $x \in A$ and $x \notin B$. Let $y \in B$, then $<y>_A \subseteq B$, $<x>_A \nsubseteq <y>_A$. If $<x>_A \subseteq <y>_A$ implies $<x>_A \subseteq B$ (since $<y>_A \subseteq B$). Thus $x \in B$ is a contradiction. If $<y>_A \subseteq <x>_A$ implies that $<y>_A \subseteq A$ (since $<x>_A \subseteq A$). Thus $B \subseteq A$ so X is chained. The converse is obvious.

2.14 Remark
A chained fuzzy module is indecomposable.

**Proof:** Suppose X is decomposable, then $X = A \oplus B$ for some fuzzy submodule A and B of X. Thus $A \cap B = \{0\}$ is a contradiction (proposition 2.9).

Now, we introduce the notion of chained fuzzy ring. First we have the following definition.

2.15 Definition, [11]
A ring R is called chained if and only if for each fuzzy ideals I, J of R either $I \subseteq J$ or $J \subseteq I$.

2.16 Definition
A fuzzy ring X of a ring R is called chained if and only if for each fuzzy ideals I, J of X either $I \subseteq J$ or $J \subseteq I$.

2.17 Remark
A fuzzy ring X is chained if and only if $X_t$ is chained ring $\forall t \in (0,1]$.

**Proof:** It is easy so it is omitted.

2.18 Definition, [8]
A fuzzy module X of an R-module M is called multiplication fuzzy module if for each nonempty fuzzy submodule A of X, there exists a fuzzy ideal I of R such that $A = IX$.

2.19 Proposition
Let X be a multiplication module of an R-module M if R is a chained ring then X is a chained fuzzy module.

**Proof:** Let A and B be fuzzy submodules of X. Then there exists fuzzy ideals I and J of R such that $A = IX$ and $B = JX$, since $I \subseteq J$, and $J \subseteq I$, ideals of R and R is chained, therefore $I_i \subseteq J_i$ or $J_i \subseteq I_i$. Thus $I_1 \subseteq J$ or $J_1 \subseteq I$ (by remark 2.3) implies that $IX \subseteq JX$ or $JX \subseteq IX$. Thus $A \subseteq B$ or $B \subseteq A$.

2.20 Definition
Let X be a chained fuzzy module of an R-module M and let $V(X) = \{(O_1;X_t) | X_t \in X\}$. 

2.21 Definition, [10]
Let X be a non-empty fuzzy module of R-module M. The fuzzy annihilator of A denoted by ($F$-annA) is defined by $\{X_t; x \in R, X_t A \subseteq O_1\}, t \in [0,1]$, where A is a proper fuzzy submodule of X.

Note that: $(F$-annA)(a) = sup $\{t \in [0,1], aX_t \subseteq O_1\}$, for all $a \in R$; that is F-annA = ($O_1$; A).

2.22 Definition, [10]
A fuzzy module X is called faithful if $F$-annX = $O_1$.

2.23 Remark
If X is a chained faithful fuzzy module then $\bigcap_{O_t = X_t \in X} (O_1; X_t) = O_1$, $X_t \in X$. 

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Proof: If \( O_{x,t} \neq O_1 \) then there is \( r_t \in R \), \( r_t \neq O_1 \) such that \( r_t x_t = O_1 \), \( \forall x_t \in X \) then \( r_t X = O_1 \) a contradiction.

2.24 Definition, [13]
A fuzzy ideal \( A \) of a ring \( R \) is called fuzzy prime ideal, if \( A \) is non-constant and for any fuzzy ideals \( B \) and \( C \) of \( R \) such that \( B \circ C \subseteq A \), then either \( B \subseteq A \) or \( C \subseteq A \).
Equivalently, A fuzzy ideal \( A \) of a ring \( R \) is called fuzzy prime ideal if \( A \) is a non-constant and for all \( a, b \) fuzzy singletons of \( R \) such that \( a \circ b \subseteq A \) implies that either \( a \subseteq A \) or \( b \subseteq A \), \( \ell, h \in [0,1] \).

2.25 Remark
If \( X \) is a chained fuzzy module of an \( R \)-module \( M \) then,
1. \( V(X) \) is a linearly ordered set of fuzzy ideals of \( R \).
2. \( P = \bigcup_{O_{x,t} \in X} (O_{1,x}) \) is a fuzzy prime ideal of \( R \).

Proof: (1) Let \( A, B \in V(X) \) then \( A=(O_{1,x}) \) and \( B=(O_{1,y}) \) for some \( x \neq O_1 \) and \( y_1 \in X \), since \( X \) is chained fuzzy module then \( X_1 \) is chained module (by theorem (2.4)) implies that \( V(X_1) \) is a linearly ordered set of ideals of \( R \) (see [12,remark (1.9)). Thus \( V(X) \) is a linearly ordered set of fuzzy ideals of \( R \).
(2) \( P = \bigcup_{O_{x,t} \in X} (O_{1,x}) \) is a fuzzy ideal of \( R \). To show that \( P \) is a fuzzy prime ideal, let \( a, b \in R \) such that \( a \circ b \in P \), then there is \( O_{1,x} \in X \) such that \( a \circ b \in (O_{1,x}) \). Then \( a \circ b = O_1 \). This implies that \( a \in (O_{1}, b \circ x) \). Now if \( b \circ x = O_1 \) then \( b \in (O_{1}, x) \). Thus \( b \in P \) and if \( b \circ x \neq O_1 \) then \( b \circ x \neq O_1 \in X \), and hence \( a \in P \).

3. Direct Sum of Chained Fuzzy Module
We turn attention to the direct sum of chained fuzzy modules.

3.1 Remark
If \( X \) and \( Y \) are two chained fuzzy modules of an \( R \)-module \( M_1 \) and \( M_2 \) respectively then \( X \oplus Y \) is not necessary chained fuzzy module of \( M_1 \oplus M_2 \) as the following example shows:

3.2 Example
Let \( X:Z_6 \longrightarrow \{0, \frac{1}{3} \} \) such that
\[
X(a) = \begin{cases} 
\frac{1}{3} & \text{if } a \in <2 > \\
0 & \text{if } a \notin <2 > 
\end{cases}
\]
Let \( Y:Z_6 \longrightarrow \{0, \frac{1}{3} \} \) such that
\[
Y(a) = \begin{cases} 
\frac{1}{3} & \text{if } a \in <3 > \\
0 & \text{if } a \notin <3 > 
\end{cases}
\]
It is clear that \( X \) and \( Y \) are chained fuzzy modules of \( Z_6 \). Hence \( X \oplus Y \) is not a chained fuzzy module of \( Z_6 \oplus Z_6 \). Since \( \exists \) \( A, B \) fuzzy submodules of \( X \oplus Y \), where
\[
A(a, b) = \begin{cases} 
\frac{1}{3} & \text{if } (a, b) \in <2 > \oplus <0 > \\
0 & \text{if } (a, b) \notin <2 > \oplus <0 > 
\end{cases}
\]
and
\[
B(a, b) = \begin{cases} 
\frac{1}{3} & \text{if } (a, b) \in <0 > \oplus <3 > \\
0 & \text{if } (a, b) \notin <0 > \oplus <3 > 
\end{cases}
\]
But $A(2,0)=\frac{1}{3}$, $B(2,0)=0$, that is $A \not\subseteq B$. Also $A(0,3)=0$, $B(0,3)=\frac{1}{3}$, that is $B \not\subseteq A$. Thus $X \oplus Y$ is not a chained fuzzy module of $Z_6 \oplus Z_6$.

3.3 Theorem

Let $X$ and $Y$ be a fuzzy modules of an $R$-modules $M_1$ and $M_2$ respectively, if $X \oplus Y$ is a chained fuzzy module of $M_1 \oplus M_2$ then $X$ is a chained fuzzy module of $M_1$ and $Y$ is a chained fuzzy module of $M_2$.

Proof: By similar proof of theorem (4.10) in [14].

Next, we shall indicate the behaviors of chained fuzzy modules under homomorphism.

3.4 Theorem

Let $X$ and $Y$ be a fuzzy modules of an $R$-modules $M_1$ and $M_2$ respectively. Let $f:X \longrightarrow Y$ be a fuzzy epimorphism. If $X$ is a chained fuzzy module, then $Y$ is a chained fuzzy module.

Proof: Let $A, B$ are fuzzy submodules in $Y$. Then $f^{-1}(A), f^{-1}(B)$ are fuzzy submodules in $X$ (remark (1.17),(2)), since $X$ is chained fuzzy module, then either $f^{-1}(A) \subseteq f^{-1}(B)$ or $f^{-1}(B) \subseteq f^{-1}(A)$.

Now, if $f^{-1}(A) \subseteq f^{-1}(B)$, then $f(f^{-1}(A)) \subseteq f(f^{-1}(B))$ (by lemma (1.8),(2)).
Similarly, if $f^{-1}(B) \subseteq f^{-1}(A)$, then $B \subseteq A$. Therefore $Y$ is a chained fuzzy module.

3.5 Proposition

Let $X$ and $Y$ be two fuzzy modules of an $R$-modules $M_1$ and $M_2$ respectively. Let $f:X \longrightarrow Y$ be a fuzzy homomorphism and every submodule of $Y$ is $f$-invariant. If $Y$ is a chained fuzzy module, then $X$ is a chained fuzzy module.

Proof: Let $A, B$ are fuzzy submodules in $X$. Hence $f(A), f(B)$ are fuzzy submodules in $Y$ (remark (1.17),(1)), since $Y$ is a chained fuzzy module then $f(A) \subseteq f(B)$ or $f(B) \subseteq f(A)$.

Now, if $f(A) \subseteq f(B)$, then $f^{-1}(f(A)) \subseteq f^{-1}(f(B))$ (by lemma (1.8),(4)). Hence $A \subseteq B$ (by lemma (1.8),(1)).
Similarly, if $f^{-1}(B) \subseteq f^{-1}(A)$, then $B \subseteq A$. Therefore $X$ is a chained fuzzy module.

References

الموديولات الضبابية المسلسلة

شروق بهجت
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الخلاصة

لتكن حلقة R حلقة أيالية ذا عنصر محايد.
في هذا البحث قدمنا مفهوم الموديولات الضبابية المسلسلة تعميماً لمفهوم الموديولات المسلسلة. لقد أعطينا العديد من التميزات والخواص الأساسية لهذا المفهوم.