MODULES WITH CHAIN CONDITIONS ON SEMISMALL SUBMODULES

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Abstract
Let R be an associative ring with identity and M be unital non zero R-module. A submodule N of a module M is called a semismall submodule of M (briefly $N \ll_{S} M$) if $N = 0$ or for each nonzeror submodule K of N, $N / K \ll M / K$. In this work, we study this kind of submodule of M and the modules which is satisfies the ascending chain condition (a. c. c.) and descending chain condition (d. c. c.) on semismall submodules. Then we generalize the Rad(M) into $s$- Rad(M). It is equale to the sum of all semismall submodule of M. We show that if N not semismall submodule of M. Then $s$- Rad (N) = $N \cap s$- Rad (M) and we discuss some of the basic properties of this types of submodules.

Introduction
Let R be a ring with identity and M be a non zero unital R-module. A submodule N of a module M is called a small submodule of M, denoted by $N \ll M$, if $N + L \neq M$ for any proper submodule L of M. In [2] Imaam and Layla introuced the definition of the concept of semismall submodule that A submodule N of a module M is called a semismall submodule of M (briefly $N \ll_{S} M$) if $N = 0$ or for each nonzeror submodule K of N, $N / K \ll M / K$. In section one, we review the concept of semismall submodule and we discuss some of the basic properties of this types of submodules. In this section, we introduce the definition of module which is satisfies the ascending chain condition (a. c. c.) and descending chain condition (d. c. c.) on semismall submodules as a
generalization of chain condition (a. c. c.) and
descending chain condition (d. c. c.) on small
submodules [3] and we study the relation between
the ring that satisfies (a. c. c.) and descending
chain condition (d. c. c.) on semismall ideals.
It is known that Rad(M) is the sum of all small
submodules of M. In section Two we generaliz
Rad(M) into s- Rad(M) as the sum of all
semismall submodules of M,
s-Rad(M) =∑{L ≤ M ,L semismall submodule of
M}.and we will study  this class  of submodules.
Let N and L be  submodules of a module M. N is
called a supplement of L in M if M=N +L and
N ∩ L<<M.[4]
In [2] If N and L be  submodules of a module M.N
is called a semi-supplement of L in M if M=N +L and
N ∩ L<<S M.
Then it is clear that every supplement submodule
is semi-supplement but the converse is not true.
We show that if N not semismall or semi-
subplement submodule of M.Then s-Rad (N) =
N ∩ s-Rad (M).

1. Modules with chain conditions on
semismall submodules
In this section , we review the concept of
semismall submodule and we discuss some of the
basic properties of this types of submodules.

M is called a semismall submodule of M (briefly N
<< S M) if N = 0 or for each nonzeror submodule
K of N, N / K <<  M / K.
It is clear that every small submodule is semismall
submodule,but the converse is not true in general for
example In Z-module Z_{12} the submodule N =< 4
> is not small but it is semismall [2].

Lemma 1.2: [ 2 ] let M be an R- module then
1- If N<< S M and K <N then K<< S M
2- If K,N are submodules of M such that
K << S N then K<< S M
3- If K<< S M and f : M→N is a homomorphism
then f ( K ) << S N
4- Let K <N < M .If K<< S M and N direct
summand of M ,then K<< S N.
5- Let M = M_1 ⊕ M_2 and let K<< M such that
K=K_1⊕K_2 then K_1<< S M_1 and K_2<< S M_2.
6- Let N <K if N<< S M and K is direct
summand then N<< S k.
The following is characterization of semismall
submodule.

M is semismall submodule of M (N << S M) iff N
+L =M for someL ≤ M implies that K+L=M for all
K ≤ N,K ≠ (0).
An R-module M is said to satisfy the ascending
chain condition (a.c.c.) on submodules.
respectively descending chin condition (d.c.c.) on
submodules if every ascending (descending) chain of
small submodules K_1⊆ K_2 ⊆ K_3 ⊆ ---- ⊆ K_n... respectively K_1 ⊇ K_2 ⊇ .... ⊇ K_n ⊇ ...
Terminates.[3].
In the following we introduce the definition of
module which satisfies the ascending chain
condition (a.c.c.) and descending chain condition
(d. c. c.) on semismall submodules

Definition (1.4): An R-module M is said to satisfy
the ascending chain condition (a.c.c.) on
semismall submodules. respectively descending
chain condition (d.c.c.) on semismall submodules if
every ascending (descending) chain of semismall
submodules K_1⊆ K_2 ⊆ K_3 ⊆ ---- ⊆ K_n... respectively K_1 ⊇ K_2 ⊇ .... ⊇ K_n ⊇ ...
Terminates.
Since every small submodule is semismall then
the following remark is clear.

Remark (1.5):If M satisfy the a.c.c.(d.c.c.) on
semismall submodules then M satisfy the
a.c.c.(d.c.c.) on small submodules.

Proposition (1.6):Let M_1 and M_2 be two
modules.Then M_1 ⊕ M_2 satisfies a.c.c.(d.c.c.) on
semismall submodules iff M_1 and M_2 satisfies
a.c.c.(d.c.c.) on semismall submodules.

Proof :Is clear by lemma (1.2).

Remark (1.7): An ideal I of a ring R is
semismall ideal if we consider R as R –module.
The following proposition is appered in [2] , here
we give it with another proof.
Proposition (1.8): Let M be a finitely generated faithful multiplication R-module, and let N = M I , for some ideal I of R then N is semismall submodule in M iff I is semismall ideal in R .

Proof : Assume N is semismall in M , and N = M I , let I + J = R , for some J of R . then M I + M J = M R = M . since N is semismall in M , then M J+K = M , K\ subseteq N , K \neq (0).
Since M multiplication R-module , then K = MT , for some ideal T of R , T \subseteq I , therefore M J+MT = M then J +T = R hence I is semismall in R .

Conversely , let N + K = M , for some submodule K of M . since M is a multiplication R-module , then K = M J , for some ideal J of R , [5]
Hence N + K = M I + M J = M (I + J) = M But M is finitely generated faithful multiplication R-module . then I + J = R since I is semismall in R , then T + J = R , T \subseteq I , T \neq (0).

Hence MT + M J = M R = M . let A = MT thus A \subseteq N and A +K = M , A \neq (0).then N is semismall submodule in M .

The following results are sequences of this proposition.

Corollary (1.9): Let M be a finitely generated faithful multiplication R-module, then R satisfies a. c. c. on semismall ideal if and only if M satisfies a. c. c. on semismall submodules .

Proof : Let N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots be an ascending chain of semismall submodule of M . Since M is a multiplication R-module, then N_i = I_i M , for some ideal I_i of R for all i , [6].
Hence M I_1 \subseteq M I_2 \subseteq M I_3 \subseteq \ldots \subseteq M I_k \subseteq \ldots . But M is finitely generated then by proposition (1.8) I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots is an ascending chain of semismall ideals in R . Since R satisfies a. c. c. on semismall ideal, then \exists K \in N , such that I_k = I_{k+1} = \ldots , hence M I_k = M I_{k+1} = \ldots . which implies N_k = N_{k+1} = \ldots , that is M satisfies a. c. c. on semismall submodules. Conversely, let I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots be an ascending chain of small ideals in R , then by proposition (1.8) M I_1 \subseteq M I_2 \subseteq M I_3 \subseteq \ldots \subseteq M I_k \subseteq \ldots is an ascending chain of semismall submodule of M .

Since M satisfies a. c. c. on semismall submodules then \exists K \in N , such that M I_k = M I_{k+1} = \ldots . But M is a finitely generated faithful module, then I_k = I_{k+1} = \ldots . [6]. Thus R satisfies a. c. c. on semismall ideals of R .

By a similar proof of cor . 1.9 ,we have .

Corollary(1.10): Let M be a finitely generated faithful multiplication R-module, then M satisfies d.c. c. on semismall submodules, if and only if R satisfies d. c. c. on semismall ideal.
Hence we have the following result :

Theorem(1.11) : Let M be a finitely generated faithful multiplication R-module, then the following are equivalent.
1) M satisfies a.c.c (d.c.c) on semismall submodules
2) R satisfies a.c.c (d.c.c) on semismall ideals .
3) S \equiv \text{End}_R(M) satisfies a.c.c (d.c.c) on semismall ideals .
4) M satisfies a.c.c (d.c.c) on semismall submodules as S-module .

Proof : (1) \Rightarrow (2) By cor (1.6)
(2) \Rightarrow (3) since M is a finitely generated faithful multiplication R-module, then R \approx S hence R satisfies a.c.c(d.c.c) S = \text{End}_R(M) satisfies a.c.c (d.c.c) on semismall ideals .
(3) \Rightarrow (4) By cor (1.9)
(4) \Rightarrow (1) By cor (1.9) S satisfies a.c.c (d.c.c) on semismall ideals . R \approx S[6] hence R satisfies a.c.c(d.c.c) on semismall ideals and by cor (1.9) M satisfies a.c. c. (d.c.c) on semismall submodules.

2. Semi radical of R-module M .

It is known that Rad(M) is the sum of all small submodules of M . In this section we introduce the semi-Rad(M) as a generalization of Rad(M).

s-Rad(M) = \sum \{ L \subseteq M , L \text{ semismall submodule of } M \} .

Since every small submodule is semismall submodules then it is clear that Rad(M) \subseteq s-Rad(M). The converse is not true, for example . In

Z-module Z_{12} , the submodules, < 4 > and < 6 > are semismall submodule , Rad(Z_{12}) = < 6 > , s-Rad(Z_{12}) = < 2 > .

The following lemmas give some properties of The s-Rad(M).

Lemma(2.1):
1- Let M be R-modul and let m \in M if Rm < s M then , m \in s-Rad(M).
2- If m \in s-Rad(M) then ther existe submodule nR \subseteq m R such that nR < s M .
3- Let $R$ be a ring and let $\varphi : M \to M'$ be a homomorphism of $R$-modules $M$, $M'$. Then $\varphi(s-Rad(M)) \leq s-Rad(M')$.

4- Let $M_{i \in I}$ be any collection of $R$-modules and let $M = \bigoplus_{i \in I} M_i$. Then $s-Rad(M) = \bigoplus_{i \in I} s-Rad(M_i)$. $5-M(s-Rad(R)) \leq s-Rad(M), R$ as an $R$-module.

Proof:

1) $mR << S M$ then , $m \in mR \leq s-Rad(M)$.

2) $m \in s-Rad(M)$ then $m=\sum n_i , n_i \in N_i << S M$ ,thus by(1.2) $Rn_i << S M \forall i$.

3) From $s-\text{Rad}(M) = \sum \{ L \leq M , L \text{ semismall submodule of } M \}$. it follows that $\varphi(s-Rad(M))= \sum_{L \leq M} \varphi(L)$, by Lemma 1.2 , $\varphi(L) << S M'$. thus $\varphi(s-Rad(M)) \leq s-Rad(M')$.

4) From (2) Then $s-\text{Rad}(M_i) \leq s-Rad(M)$ then $\sum_{i \in I} s-Rad(M_i) = \bigoplus_{i \in I} s-Rad(M_i) \leq s-Rad(M)$. Let now $m=\sum m_i \in s-Rad(M)$ and let $\beta_i : M \to M_i$ be the $i$th projection then $\beta_i(m) = m_i \in s-Rad(M_i)$ by(2).And so $m \in \bigoplus s-Rad(M_i)$ hence $s-Rad(M) \leq \bigoplus s-Rad(M_i)$.

5) Let $m \in M$, then $\varphi_m : R \to M$ defined by $\varphi_m(r) = mr$ is a homomorphism thus by (3) we have $m(s-Rad(R)) = \varphi(s-Rad(R)) \leq s-Rad(M)$ then, $\sum_{m \in M} m(s-Rad(R)) = M(s-Rad(R)) \leq s-Rad(M)$.

Remark (2.2): Let $M$ be an $R$-module then $s-Rad(M)=M$ iff all finitely generated submodules are semismall submodule of $M$.

Proof: Suppose $s-Rad(M)=M$ and $N$ a f.g submodules of $M$ with $N+L=M$ then $N=Rx_1 + Rx_2 + \ldots + Rx_n$, $x_i \in M = s-Rad(M)$ then $Rx_i << S M$ thus $K_i +L=M \forall i$ $K_i \leq N$, $K_i \neq 0$. Conversely , let $m \in M$ then $Rm$ is finitely generated then $Rm << S M$ , $m \in s-Rad(M)$.

Remark (2.3): Let $M$ be an $R$-module if $s-Rad(M) << S M$ then $M / s-Rad(M)$ has no non-zero semismall submodule .

Proof: Let $L /s-Rad(M)$ be semismall submodule of $M /s-Rad(M)$, then $L << S M$ To show that $L /s-Rad(M) + (K+ s-Rad(M) /s-Rad(M))=M /s-Rad(M)$ then $A+K+ s-Rad(M) = M , A \leq L$, $A \neq 0$. Then $L << S M$ thus $L \leq s-Rad(M)$ therefore $L= s-Rad(M)$, Then $L /s-Rad(M)$ is zero semismall submodule .

Proposition (2.4): Let $M$ be a module then $s-Rad(M)$ is Noetherian if and only if $M$ satisfies a. c. c. on semismall modules.

Proof: ($\Rightarrow$) is Clear. ($\Leftarrow$) Suppose that $s-Rad(M)$ is not Noetherian . Let $N_1 \subsetneq N_2 \subsetneq \ldots \subsetneq N_k \subsetneq \ldots$ be an infinite ascending chain of semismall submodule of $M$. Let $a_i \in N_1$ and $a_j \in N_j$ for each $i,j \geq 1$. For any $K \geq 1$, let $N_k = \sum_{j \geq 1} a_j R$. Then $N_k$ is finitely generated and $N_k \leq s-Rad(M)$ then $N_k << S s-Rad(M)$ by Lemm(1.2) and $N_1 \subsetneq N_2 \subsetneq \ldots$ thus $M$ not satisfy a. c. c. on semismall modules.

In[2] the notion of semi-supplement was introduced , and show that every supplement submodule is semi-supplement submodule.

Definition (2.5): Let $N$ and $L$ be submodules of a module $M$. $N$ is called a semi-supplement of $L$ in $M$ if $M = N + L$ and $N \leq L << S M$.

Proposition(2.6): Let $N$ be submodules of a module $M$ consider the following statements;

1) $N$ is semi-supplement submodule in $M$ .

2) $N$ is not semismall submodule in $M$.

3) For all $x \leq N, x << S M$ then $x << S N$. Then $(1) \Rightarrow (2) \Rightarrow (3)$ and $(3) \Rightarrow (1)$ if $M$ is weakly supplemented.

Proposition(2.7): Let $N$ be not semismall submodule of $M$. Then $s-Rad(N) = N \cap s-Rad(M)$.

Proof: By lemma 2.1 it is clear that $s-Rad(N) \leq M \cap s-Rad(M)$ . Now, we have to show that $N \cap s-Rad(M) \leq s-Rad(N)$ . Let $x \in N \cap s-Rad(M)$ then $x \in N$ implies that $xR \leq N$ and $x \in s-Rad(M)$ then by lemma (2.1) there exits submodule $Rn \leq Rx$ such that $Rn << S N$. Therefore $x \in s-ad(N)$ i.e. $N \cap s-Rad(M) \leq s-Rad(N)$ . Thus $s-Rad(N) = N \cap s-Rad(M)$.
Corollary (2.8): Let $M$ be an $R$-module and let $N$ be a semi-supplement submodule. Then $s\text{-Rad}(N) = N \cap s\text{-Rad}(M)$.

Proof: By prop.(2.6) and prop.(2.7).

Corollary (2.9): Let $M$ be an $R$-module and let $N$ be a direct summand of $M$. Then $s\text{-Rad}(N) = N \cap s\text{-Rad}(M)$.

Proof: Since every direct summand of $M$ is a supplement submodule and every supplement submodule is a semi-supplement submodule, then by corollary (2.8), $s\text{-Rad}(N) = N \cap s\text{-Rad}(M)$.

Lemma (2.10): Let $M$ be a module such that $s\text{-Rad}(M) \subseteq M$. Let $M_1$ and $M_2$ be direct summands of $M$ with $M_1 \leq M_2$ then $s\text{-Rad}(M_1) = s\text{-Rad}(M_2)$ if and only if $M_1 = M_2$.

Proof: Let $M = M_1 \oplus L$. Then by module low $M_2 = M_1 \oplus (M_2 \cap L)$ and by (prop. 2.1) $s\text{-Rad}(M_2) = s\text{-Rad}(M_1) \oplus s\text{-Rad}(M_2 \cap L)$. Thus if $s\text{-Rad}(M_1) = s\text{-Rad}(M_2)$ then $0 = s\text{-Rad}(M_2 \cap L) = s\text{-Rad}(M) \cap (M_2 \cap L)$ by (cor. 2.9). Thus, since $s\text{-Rad}(M) \subseteq M$, we get $M_2 \cap L = 0$ and hence $M_1 = M_2$.

References: