Criticality Importance Measure and the Increase of System Reliability through Active Redundancy
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ABSTRACT

One of the important problems in reliability is “how to design better systems”. As known, active redundancy is a technique can be used for improving system reliability “makes it better”. If the system under usage is composed of different components, then we will have different improvements in the reliability of this system when we add active redundancy to its components.

In this paper, we investigate the effect of active redundancy upon system reliability when applied at various places and in various systems. In our work, we study the problem of how to choose components for active redundancy. Some results are given, our results depend on component criticality importance also some examples are presented.

الخلاصة

إن أحد المشاكل المهمة في المعولية هي كيفية تصميم أنظمة أفضل. كما نعرف أن المجانحة الفعالة هي طريقة تستخدم لتحسين معولية نظام معين “جعله أفضل”. إذا كان النظام المستخدم يتركب من مركبات مختلفة فاننا سوف نحصل على تحسينات مختلفة في معولية هذا النظام عند إضافة وحدات إضافية إلى مركباته.

في هذا البحث، سوف نفحص تأثير المجانحة الفعالة على معولية نظام عندما تطبق في أماكن مختلفة من النظام وعندما تطبق في أنظمة مختلفة. في بحثنا سوف ندرس مسألة كيفية اختيار مركبات للمجانحة الفعالة. تم إعطاء بعض النتائج، أن نتائجنا تعتمد على "component criticality importance"
Introduction

It is well known that adding active redundancy to selected component(s) of a system is one way of improving the reliability of this system. In general, the effect of active redundancy on system reliability depends on which component is chosen. So, it is important to know where to place the redundancy.

There are two problems involved:

1. What is the increase of system reliability after active redundancy of a specific component?
2. Which component is to be chosen for active redundancy so that the increase of system reliability is largest?

In our work, we discuss the problem of increasing system reliability through active redundancy of a component, this increase is directly proportional to the component criticality importance measure $I_k$. We give some results in this concern, we derive a formula for the increase of system reliability for series and parallel systems. Some bounds are given for the increase of system reliability also, in this part we investigate the problem of how to choose components for active redundancy, and finally we give some examples concerning our discussion.

Notation

$X_k$ non negative random variable representing the lifetime of component $k$ in the system with lifetime $S$.

$F_k$ lifetime distribution of component $k$.

$F_s$ lifetime distribution of system with lifetime $S$.

$F_k$ reliability of component $k$.

$F_s$ reliability of system.

$h(F_s)$ system reliability function.
\( I_c^{(k)} \) component criticality importance measure.

\( F_s^{(k)} \) system reliability after active redundancy of component k.

\( \Delta I_s \) increase of system reliability by active redundancy of component k.

n number of components in the system.

Assumptions

1. The components are independent and the system is coherent.
2. Active redundancy of component k implies that an independent and identically distributed component is added in parallel with the existing one.
3. Active redundancy can be added to any component.

Also, in our work we consider that parallel (parallel implies a 1-out-of-n:G system). Series (series implies a 1-out-of-n:F system).

1. How to compute \( \Delta I_s \)

The system reliability function is defined by Barlow and Proschan [1, ] as:

\[
F_s(t) = h(F_2(t), F_3(t), \ldots, F_n(t))
\]

The increase of system reliability through active redundancy of component k is strongly connected with the importance measures of component, K. Shen and M. Xie [2] discussed this problem using Birnbaum importance measure. In this paper, we propose component criticality importance measure to study this problem.

The component criticality importance measure is used to determine the probability that the given component was responsible for system failure before time t [1, 2].
This measure is given by:

\[ I_{c}^{(k)}(t) = \frac{\partial F_{s}(t)}{\partial F_{k}(t)} \cdot \frac{F_{k}(t)}{F_{s}(t)} \]

This measure is time dependent. There are many other importance measures which are time independent. These measures have been studied by Barlow and Proschan [1,], Natvig [1,], Natvig [?] and Xie [4].

\[ \text{Some Results} \]

Result (1):

The increase of system reliability through active redundancy of component \( k \) is:

\[ \Delta_{k} = \bar{F}_{s} \cdot F_{k} \cdot I_{c}^{(k)} \] \( \cdots (1) \)

Proof:

By Barlow and Proschan [1,]

\[ \bar{F}_{s} = \bar{F}_{k} \cdot h(1_{k}, \bar{F}_{s}) + (1 - \bar{F}_{k}) \cdot h(0_{k}, \bar{F}_{s}) \]

System reliability after active redundancy of component \( k \) is:

\[ F_{s}^{(k)} = \bar{F}_{k} \cdot h(1_{k}, \bar{F}_{s}) + (1 - \bar{F}_{k}) \cdot h(0_{k}, \bar{F}_{s}) \]

Where, \( \bar{F}_{k} = 1 - F_{k}^{2} \)

The increase of system reliability is:

\[ \Delta_{k} = F_{s}^{(k)} - \bar{F}_{s} \]

\[ = \bar{F}_{k} \cdot h(1_{k}, \bar{F}_{s}) + (1 - \bar{F}_{k}) \cdot h(0_{k}, \bar{F}_{s}) - \bar{F}_{k} \cdot h(1_{k}, \bar{F}_{s}) - (1 - \bar{F}_{k}) \cdot h(0_{k}, \bar{F}_{s}) \]

\[ = \bar{F}_{k} \cdot h(1_{k}, \bar{F}_{s}) + h(0_{k}, \bar{F}_{s}) - \bar{F}_{k} \cdot h(0_{k}, \bar{F}_{s}) - \bar{F}_{k} \cdot h(1_{k}, \bar{F}_{s}) - h(0_{k}, \bar{F}_{s}) \]

\[ = F_{k} \cdot h(1_{k}, \bar{F}_{s}) - h(0_{k}, \bar{F}_{s}) \]

\[ = \bar{F}_{k} \cdot \frac{\partial F_{s}}{\partial F_{k}} \cdot F_{k} \]

\[ = \frac{\partial \bar{F}_{s}}{\partial F_{k}} \left( F_{k} - \bar{F}_{k} \right) \]
\[
\frac{\partial \bar{F}_z}{\partial \bar{F}_k} \cdot F_k \cdot \bar{F}_z
\]

But,
\[
I_c^{(k)}(t) = \frac{\partial \bar{F}_z(t)}{\partial \bar{F}_k(t)} \cdot \frac{F_k(t)}{\bar{F}_z(t)}
\]

So,
\[
\Delta_k = \bar{F}_z \cdot F_k \cdot I_c^{(k)}
\]

**Note (1):**

The relation \( I_c^{(i)} < I_c^{(j)} \) does not imply \( \Delta_i < \Delta_j \) as shown in example (5) in section 5.

In the next result, we derive a formula for the increase of system reliability for series systems.

**Result (7):**

If component \( k \) is in series with the rest of the system, then the increase of system reliability after active redundancy of component \( k \) is:

\[
\Delta_k = \bar{F}_z \cdot F_k \quad \cdots \cdots \quad (2)
\]

Proof:

Let \( \bar{F}_r \) be the reliability of the rest of the system.

Note that \( \bar{F}_z = \bar{F}_k \cdot \bar{F}_r \)

The component criticality importance measure of component \( k \) is:

\[
I_c^{(k)}(t) = \frac{\partial \bar{F}_z(t)}{\partial \bar{F}_k(t)} \cdot \frac{\bar{F}_k(t)}{\bar{F}_z(t)} = \bar{F}_r \cdot \frac{\bar{F}_k(t)}{\bar{F}_z(t)} = 1
\]

So, from (1) above we have

\[
\Delta_k = \bar{F}_z \cdot F_k
\]
In the following result we apply result (1) to parallel system to derive a formula for the increase of system reliability for parallel systems.

Result (1):
If component k is in parallel with the rest of the system, then the increase of system reliability after active redundancy of component k is:

$$\Delta_{ic} = F_z \cdot F_k$$ \hspace{1cm} (3)

Proof:
Let $F_r$ be the lifetime distribution of the rest of the system.

Note that $F_z = F_k \cdot F_r$

The component criticality importance measure of component k is:

$$I_c^{(k)} = \frac{\partial F_z}{\partial F_k} \cdot \frac{F_k}{F_z} = \frac{\partial F_z}{\partial F_k} \cdot \frac{F_k}{F_z} = F_r \cdot \frac{F_k}{F_z}$$

So, from (1) we have

$$\Delta_{ic} = F_z \cdot F_k$$

In result (2) below, we give bounds on $\Delta_{ic}$.

Result (2):
In a coherent system the system reliability increase through active redundancy of component k satisfies:

$$\frac{c_k}{1+c_k} \cdot F_k \cdot F_z \leq \Delta_k \leq \min(F_k \cdot F_z, F_z), \quad \text{for} \quad k = 1, r, \ldots, n$$

\hspace{1cm} ............(2)

Where $c_k \equiv F_z I_c^{(k)}$

Proof:

$$\frac{\partial F_z}{\partial F_k} \leq \min \left( \frac{F_z}{F_k}, \frac{F_z}{F_r} \right) \quad \text{by Xie [1]}$$

So, $I_c^{(k)} = \frac{\partial F_z}{\partial F_k} \cdot \frac{F_z}{F_z} \leq \min \left( \frac{F_z}{F_k}, \frac{F_z}{F_r} \right) \cdot \frac{F_k}{F_z}$
\[ \Delta_k = F_2 F_k I_c^{(k)} \leq \min \left( \frac{F_2}{F_k}, \frac{F_k}{F_k} \right) \cdot \frac{F_k}{F_2} \cdot F_k \]

\[ = \min \left( \frac{F_2}{F_k}, \frac{F_k}{F_k} \right) \cdot F_k \]

\[ = \min (F_k \cdot F_2, F_k \cdot F_k) \quad \text{........... (1)} \]

While, \( F_2 \neq 0 \),

\[ \frac{\Delta_k}{F_2} = \frac{F_2 F_k I_c^{(k)}}{F_2 I_c^{(k)} + h(0_k, F)} \geq \frac{F_2 F_k I_c^{(k)}}{F_2 I_c^{(k)} + 1} \]

\[ = \frac{c_k}{1 + c_k} \cdot F_k \]

So, \( \Delta_k \geq \frac{c_k}{1 + c_k} \cdot F_k \cdot F_2 \quad \text{........... (2)} \]

From (1) and (2), the proof is done.
How to choose components for active redundancy

In order to have largest improvement on system reliability, we must know where to add active redundancy. In our work, we consider that the cost, space, weight, etc. are constraints so that we may only have one spare part independent of which component we choose. Here, we give criterion for choosing component for active redundancy. This allocation problem have been studied by Barlow and Proshan \[1,2\], Messinger et al. \[3\], Misra et al. \[4\], and Misra et al. \[5\]. But in this work we try a new direction.

Criterion

The component which gives largest increase in system reliability through active redundancy is that which must be chosen for active redundancy i.e., we choose a component \(k\) where

\[
K: \Delta_k = \max \Delta_i, \; i = \frac{1}{r}, \ldots, n
\]

By this criterion the most important component is the one which has the lowest reliability, since the component which has very high reliability would have smaller improvement potential compared with a less reliable one. In other words, the utility of a more reliable component has been exerted to a fuller extent so there is not much more room left for improvement.

We consider active redundancy when applying the previous criterion.

Result (\(\varphi\):

For a series (parallel) system, the component to be chosen for active redundancy should be the one which possess the largest \(F_k(\bar{F}_k)\).
7. Examples

Example 1-
The system in figure (1) from [7],

Let,
\[ F_1 = 0.90 \]
\[ F_2 = 0.91 \]
\[ F_3 = 0.90 \]
\[ F_4 = 0.91 \]
\[ F_5 = 0.90 \]

By applying result (1) which component must be chosen to increase the reliability.

Solution:
\[ F_s = (1 - F_1 F_2 F_3) F_4 + F_4 F_5 (1 - F_1 F_3) \]
\[ = 0.999914 \]

\[ \Delta_1 = F_s^{(1)} - F_s = 2.73 \times 10^{-4} \]
\[ \Delta_2 = F_s^{(2)} - F_s = 1.77 \times 10^{-4} \]
\[ \Delta_3 = F_s^{(3)} - F_s = 2.73 \times 10^{-4} \]
\[ \Delta_4 = F_s^{(4)} - F_s = 2 \times 10^{-3} \]
\[ \Delta_5 = F_s^{(5)} - F_s = 1.9 \times 10^{-3} \]

So, we first choose component \( \xi \) for active redundancy.

Now, the components criticality importance measures are:

\[ I_c^{(1)} = \frac{\partial F_s}{\partial F_1} \cdot \frac{F_1}{F_s} = 5.4655 \times 10^{-3} \]
\[ I_c^{(2)} = \frac{\partial F_s}{\partial F_2} \cdot \frac{F_2}{F_s} = 2.2130 \times 10^{-3} \]
\[ I_c^{(3)} = \frac{\partial F_s}{\partial F_3} \cdot \frac{F_3}{F_s} = 5.5 \times 10^{-3} \]
\[ I_c^{(4)} = \frac{\partial F_s}{\partial F_4} \cdot \frac{F_4}{F_s} = 5.023 \times 10^{-2} \]
\[ I_c^{(5)} = \frac{\partial F_s}{\partial F_5} \cdot \frac{F_5}{F_s} = 3.7992 \times 10^{-2} \]

As seen, according to component criticality importance measure component \( \xi \) must be chosen first for active redundancy i.e., there is an agreement between our criterion and the component criticality importance measure.

After active redundancy of component \( \xi \), the system reliability is increased from \( \Delta_t \) to \( \Delta_t' \). From (\( \text{Eq} \)), we can have bounds on \( \Delta_t \) as: \( 0.0019 \leq \Delta_t \leq 0.0022 \)

Example - \( \text{Eq} \)
Consider the following simple structure [\( t \) \( \text{Eq} \)]
Let,
\[ F_1 = 0.9 \]
\[ F_2 = 0.9 \]
\[ F_2 = 0.9 \]

By applying result (1) which component must be chosen to increase the reliability.

Solution:
\[ F_s = 1 - (1 - F_1 F_2)(1 - F_3) \]
\[ = 0.9973 \]
\[ \Delta_1 = F_s^{(1)} - F_s = 1.311 \times 10^{-3} \]
\[ \Delta_2 = F_s^{(2)} - F_s = 2.098 \times 10^{-3} \]
\[ \Delta_3 = F_s^{(3)} - F_s = 3.7 \times 10^{-3} \]

Therefore, component \( r \) should be chosen first for active redundancy.

Now, the components criticality importance measures are:
\[ I_c^{(1)} = \frac{\partial F_s}{\partial F_1} \cdot \frac{F_1}{F_s} = 2.632 \times 10^{-2} \]
\[ I_c^{(2)} = \frac{\partial F_s}{\partial F_2} \cdot \frac{F_2}{F_s} = 2.632 \times 10^{-2} \]
\[ I_c^{(3)} = \frac{\partial F_s}{\partial F_3} \cdot \frac{F_3}{F_s} = 12.3 \times 10^{-2} \]
As seen, there is an agreement between our criterion and the component criticality importance measure.

After active redundancy of component $r$, the system reliability is increased from $0.9977$ to $0.99994$. From (7), we can have bounds on $\Delta_4$ as: $0.00326 \leq \Delta_4 \leq 0.00366$

The agreement between our criterion and the component criticality importance measure isn’t always observed as seen in the next example.

**Example - $r$**

Consider the following $r$-out-of-$r$ structure

![Diagram of r-out-of-r structure]

Let,

$F_1 = 0.97$

$F_2 = 0.99$

$F_2 = 0.98$

Applying note (7) above.

**Solution:**
\[ F_s = F_1 F_2 + F_1 F_3 + F_2 F_3 - 2F_1 F_2 F_3 \]
\[ \Delta_1 = F_s^{(1)} - F_s = 3.31 \times 10^{-2} \]
\[ \Delta_2 = F_s^{(2)} - F_s = 2.99 \times 10^{-3} \]
\[ \Delta_3 = F_s^{(3)} - F_s = 1.87 \times 10^{-3} \]

Notice that, \( F_1 \leq F_2 \leq F_3 \)

According to Barlow and Proschan \[1\], we know that if \( F_i \geq \frac{1}{2} \), \( i = 1, \, r, \, r \), then

\[ \frac{\partial F_s}{\partial F_1} \leq \frac{\partial F_s}{\partial F_2} \leq \frac{\partial F_s}{\partial F_3} \]

So, if \( F_i \geq \frac{1}{2} \), \( i = 1, \, r, \, r \), then

\[ \frac{\partial F_s}{\partial F_1} \cdot F_3 \leq \frac{\partial F_s}{\partial F_2} \cdot F_3 \leq \frac{\partial F_s}{\partial F_3} \cdot F_3 \]

Which means that, \( \ell_c^{(1)} \leq \ell_c^{(2)} \leq \ell_c^{(3)} \)

That is the component with highest reliability is the most important one. But, by our criterion the most important component is the one which has the lowest reliability.
References


