فضاءات الرص من النوع - L

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الخلاصة

الغرض من هذا البحث دراسة أنواع جديدة من التراس في الفضاءات التilogيه الثنائية، قد نقدم التراس من النوع - L.
L- compact Spaces

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Abstract

The purpose of this paper is to study a new types of compactness in bitopological spaces. We shall introduce the concepts of L- compactness.

Introduction

The concept of bitopological space was initiated by Kelly[1]. A set $X$ equipped with two topologies $\tau_1$ and $\tau_2$ is called a bitopological space denoted by $(X, \tau_1, \tau_2)$.

By a directed set we mean a pair $(A, \geq)$ consisting of a non-empty set $A$ and a binary relation $\geq$ defined on $A$ and satisfies the following conditions:

1. $a \geq a$ for each $a \in A$.
2. If $a \geq b$ and $b \geq c$, then $a \geq c$ for each $a, b, c$ in $A$.
3. For each two members $a$ and $b$ of $A$, there exists a member $c \in A$ such that $c \geq a$ and $c \geq b$.

If $(A, \geq)$ is a directed set and $f$ is a function of $A$ into a non-empty set $X$, then $f$ is called a "net" in $X$ and is denoted by $(f, X, A, \geq)$. The image of $a \in A$ under $f$ is denoted by $f_a$ and a net in $X$ will be sometimes denoted by $\{f_a: a \in A\}$. [2]

A "filter" on a non-empty set $X$ is a non-empty family $F$ of subsets of $X$ with the following properties:

1. $\emptyset \not\in F$.
2. If $F \in F$ and $F \subseteq H$, then $H \in F$.
3. If $F \in F$ and $H \in F$, then $F \cap H \in F$.

A filter on a non-empty set is said to be an ultrafilter if and only if it is not properly contained in any other filter on this set. [2]
L-open set was studied by Al-swid[2], a subset G of a bitopological space \( (X, \tau_1, \tau_2) \) is said to be “L-open” set if and only if there exists a \( \tau_1 \)-open set U such that \( U \subseteq G \subseteq \text{cl}_{\tau_2}(U) \). The family of all L-open subsets of X is denoted by L-O(X). The complement of an L-open set is called “L-closed” set. The family of all L-closed subsets of X is denoted by L-C(X). In a bitopological space \( (X, \tau_1, \tau_2) \) every \( \tau_1 \)-open set is an L-open set[3]. The union of any family of L-open subsets of X is an L-open set, but the intersection of any two L-open subsets of X need not be L-open set[2]. Al-Talkahny [3] introduced two new concepts “L-2\(-T\)2-spaces” and “L-continuous functions”. A bitopological space \( (X, \tau_1, \tau_2) \) is said to be “L-2\(-T\)2-space” if and only if for each pair of distinct points x and y in X, there exist two disjoint L-open subset G and H of X such that \( x \in G \) and \( y \in H \). Let \( \left( X, \tau_1, \tau_2 \right) \) be any bitopological spaces and let \( f : X \rightarrow Y \) be any function, then f is said to be “L-continuous” function if and only if the inverse image of any L-open subset of Y is an L-open subset of X.

2- L-compactness

Definition(2.1)

Let \( (X, \tau_1, \tau_2) \) be a bitopological space and let A be a subset of X. By an “L-open cover of A” we mean a subcollection of the family L-O(X) which covers A.

Remark(2.2):

Every \( \tau_1 \)-open cover in a bitopological space \( (X, \tau_1, \tau_2) \) is an L-open cover.

The converse of remark (2.2) is not true in general as the following example shows:

Example (2.3)

\[
X = \{1, 2, 3, 4\} \\
\tau_1 = \{X, \phi, \{1\}, \{2, 3\}, \{1, 2, 3\}\} \\
\tau_2 = \{X, \phi, \{1\}\} \\
F_2 = \{X, \phi, \{2, 3, 4\}\} \\
L-O(X) = \{X, \phi, \{1\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 4\}, \{1, 2\}, \{1, 3, 4\}, \{1, 3\}, \{2, 3, 4\}\}_{L} \\
C = \{\{1\}, \{2, 3, 4\}\}, \text{ note that C is an L-open cover of X, but it is not } \tau_2 \text{-open cover.}
\]

Definition(2.4)

A bitopological space \( (X, \tau_1, \tau_2) \) is said to be “L-compact space” if and only if every L-open cover of X has a finit subcover.

Proposition (2.5)
If a bitopological space $(X, \tau_1, \tau_2)$ is an L-compact space, then $(X, \tau_1)$ is a compact space.

Proof: follows from remark (2.2).

**Remark (2.6)**

The opposite direction of proposition (2.5) is not true in general, as the following example shows:

Let $X = \mathbb{N}$ and let $x_o \in \mathbb{N}$

$\tau_1 = \{\mathbb{N}, \phi, \{x_o\}\}$

$\tau_2 = I =$ The indiscrete topology

$L - O(X) = \{U \subseteq \mathbb{N}; x_o \in U \text{ or } U = \phi\}$

Note that $(\mathbb{N}, \tau_1)$ is compact but $(\mathbb{N}, \tau_1, \tau_2)$ is not L-compact.

**Proposition (2.7)**

An L-closed subset of an L-compact space is L-compact.

Proof:

Let $A$ be an L-closed subset of an L-compact space $(X, \tau_1, \tau_2)$ and let $\{G_\alpha : \alpha \in \Lambda\}$ be an L-open cover of $A$. Then $\{G_\alpha : \alpha \in \Lambda\} \cup A^c$ forms an L-open cover of $X$ which is L-compact space. So there are finitely many elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X = \bigcup_{i=1}^n G_{\alpha_i} \cup A^c$, it follows that $A \subseteq \bigcup_{i=1}^n G_{\alpha_i}$. Hence $A$ is an L-compact.

**Corollary (2.8)**

An L-closed subset of an L-compact space $(X, \tau_1, \tau_2)$ is $\tau_1$-compact.

Proof:

Follows from proposition (2.7) and (2.5).

**Corollary (2.9)**

A $\tau_1$-closed subset of an L-compact space $(X, \tau_1, \tau_2)$ is L-compact.

Proof:

Since every $\tau_1$-closed set is an L-closed set and by proposition (2.7).

**Corollary (2.10)**

A $\tau_1$-closed subset of an L-compact space $(X, \tau_1, \tau_2)$ is $\tau_1$-compact.
Proof:
Follows from corollary (2.9) and proposition (2.5).

**Proposition (2.11)**

The L-continuous image of an L-compact space is an L-compact.

Proof:

Suppose that \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2) \) is an L-continuous and onto function and \( X \) is an L-compact space. Let \( \{G_x: x \in A\} \) be an L-open cover of \( Y \),

it follows that \( \{f^{-1}(G_x): x \in A\} \) is an L-open cover of \( X \) which is L-compact. So there are finitely many elements \( x_1, x_2, \ldots, x_n \) such that

\[
X - \bigcup_{i=1}^{n} f^{-1}(G_{x_i}) = f^{-1} \left( \bigcup_{i=1}^{n} G_{x_i} \right)
\]

Therefore \( Y = \bigcup_{i=1}^{n} G_{x_i} \), hence \( Y \) is an L-compact.

**Corollary (2.12)**

Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2) \) be an L-continuous function, then \( f(A) \) is a compact subset of \( (Y, \tau'_1) \) for each L-compact subset \( A \) of \( X \).

Proof:

Follows from propositions (2.11) and (2.5).

It is known that every compact subset of any \( T_2 \) -space is closed. If we change the concepts of compact, \( T_2 \) and closed by the concepts L-compact-\( T_2 \) and L-closed, then this fact being invalid in general, as the following example shows:

**Example (2.13)**

\[
X = \{1, 2, 3\} \\
\tau_1 = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\} \\
\tau_2 = I \\
L-O(X) = \{X, \phi, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\} \\
L-C(X) = \{X, \phi, \{2, 3\}, \{1, 3\}, \{3\}, \{2\}, \{1\}\} .
\]

Clear that \( X \) is an L-\( T_2 \)-space. If \( A = \{1, 2\} \), then \( A \) is an L-compact subset of \( X \), but it is not L-closed.

**Definition (2.14)**: [3]

Let \( (X, \tau_1, \tau_2) \) be a bitopological space and let \( A \) be a subset of \( X \), \( x \in X \). Then \( A \) is called an
L-neighborhood of x if and only if there is an L-open set G in X such that \( x \in G \subseteq A \).

**Definition (2.15) [3]**

Let \( (X, \tau_1, \tau_2) \) be a bitopological space and let A be a subset of X. The intersection of all L-closed set containing A is called “L-closure of A” denoted by L-cl(A).

**Theorem (2.16) [4]**

Let \( (X, \tau_1, \tau_2) \) be a bitopological space and let A be a subset of X. A point x in X is an L-closure point of A if and only if every L-open neighborhood of x intersects A.

**Definition (2.17) [4]**

Let \( (X, \tau_1, \tau_2) \) be a bitopological space and let \( (f, X, A, \geq) \) be a net in X, then f is said to be “L-convergent ” to a point \( x_o \) in X if and only if for each L-open neighborhood N of \( x_o \), there exists an element \( a_o \in A \) such that \( f_{a_o} \in N \) for each \( a \geq a_o \).

**Definition (2.18) [4]**

Let \( (X, \tau_1, \tau_2) \) be a bitopological space and let \( (f, X, A, \geq) \) be a net in X. A point \( x_o \) in X is called an “L-cluster point of f” if and only if for each \( a \in A \) and for each L-open neighborhood N of \( x_o \), there exists an element \( b \geq a \) in A such that \( f_{b} \in N \).

**Theorem (2.19) [4]**

Let \( (X, \tau_1, \tau_2) \) be a bitopological space and let \( (f, X, A, \geq) \) be a net in X. For each \( a \in A \) let \( M_a = \{f(x) : x \geq a \in A\} \), then a point p of X is an L-cluster point of f if and only if \( p \in L-\text{cl}(M_a) \) for each \( a \in A \).

**Definition (2.20)**

Let \( (X, \tau_1, \tau_2) \) be a bitopological space and let F be a filter on X. A point x in X is called an “L-cluster point of F” if and only if each L-open neighborhood of x intersects every member of F.

**Theorem (2.21)**

Let \( (X, \tau_1, \tau_2) \) be a bitopological space and let F be a filter on X. A point p in X is an L-cluster point of F if and only if \( p \in L-\text{cl}(F) \) for each \( F \in F \).

Proof: the “first direction”

Suppose that p is an L-cluster point of F, then for each L-open neighborhood G of p, \( G \cap F \neq \emptyset \) for each \( F \in F \), it follows by theorem (2.16) that \( p \in L-\text{cl}(F) \) for each \( F \in F \).
The “second direction”

Assume that \( p \in L-cl(F) \) for each \( F \in \mathcal{F} \), then by theorem (2.16) every \( L \)-open neighborhood of \( p \) intersects \( F \) for each \( F \in \mathcal{F} \). Hence \( p \) is an \( L \)-cluster point of \( F \).

**Definition (2.22) [2]**

A collection of sets is said to have the finite intersection property (FIP) if and only if the intersection of each finite subcollection of it is non empty.

**Remark (2.23) [2]**

Every filter in a non-empty set \( X \) has the FIP.

**Theorem (2.24) [3]**

Let \( A \) be a non empty collection of subsets of a set \( X \) such that \( A \) has the FIP. Then there exists an ultra filter \( \mathcal{F} \) containing \( A \).

**Proposition (2.25) [4]**

Let \( A \) be a subset of a bitopological space \((X, \tau_1, \tau_2)\). Then \( A \) is an \( L \)-closed set if and only if \( A = L-cl(A) \).

**Theorem (2.26)**

Let \((X, \tau_1, \tau_2)\) be a bitopological space. Then the following statements are equivalent:

1- \( X \) is an \( L \)-compact space,

2- Every collection of \( L \)-closed subsets of \( X \) with the FIP has a non empty intersection, and

3- Every filter on \( X \) has an \( L \)-cluster point.

**Proof:**

1 \( \Rightarrow \) 2

Let \( \{ F_\alpha : \alpha \in \Lambda \} \) be a collection of \( L \)-closed subset of \( X \) with the FIP. suppose that \( \bigcap_{\alpha \in \Lambda} F_\alpha = \phi \), it follows by De-Morgan Laws that \( \bigcup_{\alpha \in \Lambda} F_\alpha^c = X \) therefore \( \left\{ F_\alpha^c : \alpha \in \Lambda \right\} \) forms an \( L \)-open cover for \( X \) which is an \( L \)-compact space, then there exists finitely many elements \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that \( \bigcup_{i=1}^n F_{\alpha_i}^c = X \). Again by De-Morgan Laws we have that \( \bigcap_{i=1}^n F_{\alpha_i} = \phi \) which is a contradiction since \( \{ F_\alpha : \alpha \in \Lambda \} \) has the FIP. Hence \( \bigcap_{\alpha \in \Lambda} F_\alpha = \phi \).
Let $F$ be a filter on $X$, then by remark (2.23) $F$ has the FIP, it follows that the collection \{\(L-\text{cl}(F)\): $F \in F$\} of $L$-closed subsets of $X$ also has the FIP, so by (2) there exists at least one point $x \in \bigcap \{L-\text{cl}(F)\} : F \in F$ then by theorem (2.21) $x$ is an $L$-cluster point of $F$. Thus every filter on $X$ has an $L$-cluster point.

Assume that every filter on $X$ has an $L$-cluster point and let $\mathcal{A}$ be an $L$-open cover of $X$. Suppose, if possible, $\mathcal{A}$ has no finite subcover the collection \{\(X-G\): $G \in Y$\} has the FIP, for if there is a finite subcollection \{\(X-G_i\): $1 \leq i \leq n$\} of such that \(\bigcap \{X-G_i\} = \emptyset\) this implies that \(\bigcup \{G_i\} = X\) which contradicts our supposition that $\mathcal{A}$ has no finite subcover, thus must have the FIP, it follows by theorem (2.24) that there exists an ultrafilter $F$ on $X$ containing $x \in X$, then by theorem (2.21) $x \in L-\text{cl}(F)$ for each $F \in F$, in particular $x \in L-\text{cl}(X-G)$ for each $G \in \mathcal{A}$. But $X-G$ is an $L$-closed subset of $X$ for each $G \in \mathcal{A}$, therefore by proposition (2.25) $L-\text{cl}(X-G) = X-G$ for every $G \in \mathcal{A}$. This implies $x \in \bigcap \{X-G\} : G \in \mathcal{A}\}$, so by De-Morgen Laws $x \in X-\bigcup \{G\} : G \in \mathcal{A}\}$, that is, $x \notin \bigcup \{G\} : G \in \mathcal{A}\}$, which is a contradiction with the fact that $\mathcal{A}$ is an $L$-open cover of $X$, hence $\mathcal{A}$ must have a finite subcover and consequently $X$ is an $L$-compact space.

Proposition (2.27):
Let $(X, \tau_1, \tau_2)$ be a bitopological space. If $X$ is an $L$-compact space, then every net in $X$ has an $L$-cluster point.

Proof:
Let $(f, x, A, \leq)$ be a net in $X$. For each $a \in A$ let $K_a = \{f_x : x \geq a \text{ in } A\}$. Since $A$ is directed by $\geq$, so the collection \{\(K_a\) : $a \in A$\} has the FIP. Hence \{\(L-\text{cl}(K_a)\) : $a \in A$\} also has the FIP, it follows by theorem (2.26) \(\bigcap \{L-\text{cl}(K_a)\} \neq \emptyset\) let $p \in \bigcap \{L-\text{cl}(K_a)\}$, then $p \in L-\text{cl}(K_a)$ for each $a \in A$, so by theorem (2.19) $p$ is an $L$-cluster point of $f$.

References