Processing of Near Singular Integrals in 3D Boundary Elements Method

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Abstract
In this work, the efficiency of double Gauss quadrature method, used to integrate over a rectangular element in 3D BEM, has been investigated. The efficiency of a quadrature or integration scheme is investigated by estimating the critical ratio for which the absolute relative error of the numerical integration is less than 1x10^-6. As small as the critical ratio is, the quadrature is more efficient. Also, special transformation techniques have been introduced and used to increase the accuracy and efficiency of double Gauss quadrature especially for near singular cases, where the source point is very close to the element under consideration. Three types of kernels were considered, weak, strong and hyper singular kernels which can be encountered in the integral equation of 3D elastodynamics BEM problems.

Keywords: Boundary Elements Method, near-singular integrals, Gauss quadrature, variable transformation.

1. Introduction
The Boundary Element Method (BEM) or the Boundary Integral Equation (BIE) method is a powerful technique for solving partial differential equations. It requires discretization only on the boundary of the domain and, hence, reducing the dimensionality by one [1, 2]. One of the...
biggest challenges in the BEM is the evaluation of integrals over the discretized boundary, especially in 3D problems where the integral must be performed over an area. In fact, the accuracy and efficiency of the BEM technique depends mainly on the evaluation of these integrals, in particular, the evaluation of near singular integrals which occur when the field point is very close to the area of integration.

Lachat and Watson [3] proposed an adaptive element subdivision technique using an error estimator for the numerical integration. Later, Pinder [4] proposed a regularization procedure, to be used together with the variable transformation approach to further weaken the near-strong singular integrals. A different approach using quadratic and cubic variable transformations in order to weaken the near singularity before applying Gauss quadrature was introduced by Telles [5].

Special weight function formulae for the 3D kernel $1/r^2$ were developed by Cristescu and Lebognac [6] for triangular and square plane elements. Aliabadi et al. [7, 8] introduced the subtraction of singularity method, which is based on series expansion of the fundamental solution, shape function and the Jacobian in 3D BEM problems. The reader who is interested in further details is referred to Ref. [9]. On the other hand, analytical integration can be found very efficient in this task, since it is not only useful for near singular cases but also when the source point lies on the element itself. However, it is limited only to the planar elements or polyhedral domains as can be

seen in [10]. Fayyad [11] applied the variable transformation method to evaluate near singular integrals over curved boundaries. His work was extension to previous works executed by him and other co-workers [12]-[15]. It includes using of system coordinates transformation, from Cartesian into polar coordinates, in order to weaken the singularity, then by using variable transformation for both the radius and angle, the singularity is further weakened by multiplication by the derivative, or in other terms Jacobian, of the new transformation.

In this work, the variables transformation method, called $T^p$, is applied for both the two variables of integration in rectangular-shaped elements. The singularity is efficiently weakened by the Jacobian of transformation resulting in enhanced accuracy and ability to process near singular cases. The technique can easily be extended to triangular elements. In section 2, the problem of near singular integrals will be described for 3D BEM in elastodynamics. Gauss quadrature over an area and variable transformation technique will be presented in section 3. The proposed integration scheme will be demonstrated in section 4, while investigation for different orders of singularities will be given in section 5. Finally, section 6 will be devoted to discuss the results and record some conclusions.

2. Problem Statement

Reference will be made to elastodynamics BEM problems for being the problems with the highest order of singularity. The
The fundamental solution of the displacement (also called Green's function) for 3D elastodynamics problems is given by [18]:

$$G(r) = \frac{1}{4\pi r} \left( \frac{\mathbf{I}}{r^2} \right) \left( \frac{2r \otimes \mathbf{r}}{r^3} - \frac{1}{r} \right) + \frac{3 \mathbf{r} \otimes \mathbf{r}}{r^5}$$

$$= \mathbf{H} \left( \frac{r - \mathbf{r}}{c_1} \right) - \mathbf{H} \left( \frac{r \otimes \mathbf{r}}{c_2} \right) + \frac{3 \mathbf{r} \otimes \mathbf{r}}{r^5}$$

$$\left\{ \frac{1}{c_1^2} \delta \left( \frac{r - \mathbf{r}}{c_1} \right) - \frac{1}{c_2^2} \delta \left( \frac{r - \mathbf{r}}{c_2} \right) \right\}$$

$$+ \frac{3}{r^5} \mathbf{r} \otimes \mathbf{r}$$

(1)

Where $\mathbf{x}$ is the source (field) point, $y$ is any point in the space, $t$ is the time, $r = \mathbf{x} - \mathbf{y}$, $\mathbf{I}$ is the identity matrix, $c_1, c_2$ are the speeds of pressure and shear waves respectively, $\mathbf{H}$ is the Heaviside (unit step) function and $\delta$ is the Dirac delta function. By examining the above equation, three orders of singularities have to be dealt with, weak $1/r$, strong $1/r^2$ and hyper-singular $1/r^3$ when integrating over space. Integration over time has no effect on the singularity order of the kernel. When the source point is far away enough from the element under consideration, double Gauss quadrature can be found very efficient and accurate. However, when the source point is very close to the integration element, or in other words, the nearest distance between the source point and the element is very small as compared to the largest dimension in the element, error will arise unless a special treatment is adopted especially for higher order singularities. This is due to the fact that, when $r$ becomes very small (when trying to integrate over the points close to the source point), the resulting kernel will be unstable and, hence, must be represented by a higher degree polynomial. Therefore, a higher order Gauss quadrature must be used to implement the integration which leads to long computational time.

Introducing a dimensionless ratio $R$, this represents the ratio of minimum distance between the source point and the element $(d)$ to the largest dimension $(a)$:

$$R = \frac{d}{a}$$

(2)

The efficiency of any integration scheme can be investigated by examining its ability to maintain the error within a specified limit as $R$ becomes very small. The value of $R$ for which the relative error of numerical integration is just about to exceed the tolerance is called critical ratio for that integration scheme. The tolerance is selected as $1 \times 10^{-6}$ in this work which is quite sufficient for most applications.

3. Gauss Quadrature over a Rectangle

Gauss quadrature can be combined to integrate over an area in 3D BEM [2, 17]. For a rectangular element it takes the following form:

$$\int_0^1 \int_0^1 f(\xi_1, \xi_2) d\xi_1 d\xi_2 = \sum_{n=1}^{n_a} \sum_{n=1}^{n_b} f(\eta_{n_1}, \eta_{n_2}) \omega_{n_1} \omega_{n_2}$$

(3)
Where $\zeta_1, \zeta_2$ are the coordinate coordinates, $N_1, N_2$ are the nodes of the element, and $u_1, u_2$ are the absolute and weights for the period $[0, 1]$. Although this integration scheme is efficient and accurate, when the source point is far enough from the element, it becomes erroneous for nearly singular cases unless special treatment is adopted. In this work, a special variable transformation is tested and applied to enhance the ability of double Gauss quadrature in numerical integration even when the ratio $R$ is very small. The used transform, called $T^\beta$, where $\beta$ is degree of the transform, takes the following form:

$$T^\beta(x) = (x + R)^{-\beta}$$

(4)

Which represents the reciprocal of $\beta$-order root. Applying this transformation for both curve-linear variables $\zeta_1$ and $\xi_2$ yields:

$$\hat{\zeta}_1 = T^\beta(\zeta_1) = (\zeta_1 + R)^{-\beta}$$

$$\Rightarrow \zeta_1 - \hat{\zeta}_1 = \frac{-R}{R + \zeta_1} \zeta_1$$

$$d\zeta_1 - \hat{\zeta}_1 d\hat{\zeta}_1$$

so,$$ \int_{\zeta_1}^{\xi_1} \int_{\zeta_2}^{\xi_2} f(\hat{\zeta}_1, \hat{\zeta}_2) d\hat{\zeta}_1 d\hat{\zeta}_2 =$$

$$\int_{\zeta_1}^{\xi_1} \int_{\zeta_2}^{\xi_2} f(\hat{\zeta}_1, \hat{\zeta}_2) d\hat{\zeta}_1 d\hat{\zeta}_2 =$$

$$[f(\hat{\zeta}_1, \hat{\zeta}_2) \sum_{n=1}^{n_1} \frac{\hat{\eta}_n}{\hat{\eta}_n} (\hat{\hat{\xi}}_1 - \zeta_1)]$$

$$[f(\hat{\xi}_1, \hat{\xi}_2) \sum_{n=1}^{n_2} \frac{\hat{\eta}_n}{\hat{\eta}_n} (\hat{\hat{\xi}}_2 - \zeta_2)]$$

$$w_{1a} w_{2a}$$

(5)

where $f(\hat{\zeta}_1, \hat{\zeta}_2) = f(\hat{\zeta}_1, \hat{\zeta}_2 - R, \hat{\zeta}_2 - R)$

$$\hat{\xi}_1 = \hat{\xi}_1 - R, \hat{\xi}_2 = \hat{\xi}_2 - R$$

$$\hat{\eta} = \hat{\eta} - T^\beta(\eta)$$

It is worthy to mention here that the Jacobian of the transformation $\hat{\zeta}_1, \hat{\zeta}_2$ yield very small value near singular points and, hence, will efficiently weaken the singularity. This technique will be used in combination with the following scheme to enhance the ability of Gauss quadrature to integrate over near singular elements.

4. The Proposed Integration Scheme

In order to study the effect of a small ratio $R$ and to demonstrate the idea of the proposed scheme, a rectangular planar element is considered as shown in Figure (1) for the sake of simplicity. However, the procedure can also be extended to the curved elements. Let $\xi$ be the source point; and $\xi_i$ the point on the element nearest to $\xi$. Hence, the minimum distance is simply $d = |\xi - \xi_i|$. It is clear that when the vector of minimum distance $d$ is parallel to the normal from point $\xi_i$, or in other words the point $\xi$ lies inside the element, the singularity will be higher for the same value of $d$. This is due to the fact that there will be more singular values for the integrand around $\xi_i$ with worst case when $\xi$ lies exactly on the center of the element. For that reason, we will consider the worst case for $\xi$ in this study. When $d$ is very small compared to the size of the element, element subdivision must be performed at the point $\xi_i$, as shown in Figure (1), when it is far enough from any edge of the element.
This subdivision is useful from two aspects. Firstly, it introduces smaller elements and, hence, higher ratio \( R \) which will result in better accuracy, and secondly, it permits the application of the proposed scheme which depends on the fact that the nearest point \( \xi \) is very close or lies exactly on the corner of the element.

![Figure 1](image1)

Figure (1) Rectangle Subdivision

When \( \xi \) is very close to any edge of the element, then there is no need to subdivide the adjacent edge. For example, when the distance between \( \xi \) and the edge \( x = x_i \) is less than \( 5\delta \), then there is no need to subdivide the edge \( y = y_j \), where \( (x_i, y_j) \) is the corner nearest to \( \xi \). In order to apply the proposed scheme for the task of numerical integration, the following procedure is introduced:

1. Find the nearest point \( \xi_2 \) and the minimum distance \( d \), this can be done by applying Newton-Raphson method.

2. If \( |x(\xi_2) - x_j| < 5\delta \) for any edge, then no need to subdivide in the \( x \)-direction, see Figure (2), but a new minimum distance \( d_2 \) must be considered in determination of \( R \), defined as

\[
d_2 = d - 0.5|x(\xi_2) - x_j|
\]

to compensate for loss of accuracy due to presence of more singular points around \( \xi_2 \), the same thing can be said about \( y \)-axis.

3. In case of only one subdivision in one direction is done, consider \( d_2 \) for the two resulting sub-elements. In case of no subdivision is performed, consider

\[
d_2 = d - 0.5|x(\xi_2) - y_j|
\]

for the main element. In case of full sub-division, consider \( d \) for all the resulting sub-elements.

4. Now calculate the ratio \( R \) for each (sub.)-element by dividing the (corrected) minimum distance by the largest dimension in that (sub.)-element. Then apply the appropriate transformation and Gauss quadrature scheme which can be known from the following discussion.

![Figure 2](image2)

Figure (2) Subdivision in \( y \)-direction

Note that, all the calculations in steps 1-4 can be executed in terms of curve-linear coordinates, but \( d \) must be normalized by the largest dimension in the element. This is useful when dealing with curved elements. These steps will take very little execution time (less than 1%) in comparison to the upcoming processing.
5. Investigation of the Proposed Scheme

In order to investigate the efficiency of $T^i_f$ transform as well as direct integration, a standard rectangle $(x, y) \in \mathbb{R}, 0 < x, y < 1$ was considered with a source point of coordinates $(0, 0, c_d)$, i.e., $\tilde{C}$ lies exactly on the corner of the element located at the origin. The following procedure is used to calculate the critical ratio $R_c$ for direct and transformed integration variables with different degrees of transformation, $\beta$, for three types of singularity. Both the exact (analytical) and the numerical solutions are expressed in terms of the ratio $R$ for different types of kernels, then the absolute error function $E_c$ given in eq.(6) below is plotted against $R$. The critical value of $R$ is the point at which $E_c$ is just less than 1$\times$10$^{-6}$. Figure (3) shows two cases of the plot $E_c - R$ corresponding to two different quadratures. The oscillation of $E_c$ is reasoned to the fact that Gauss Quadrature is a polynomial of the integration variables and, hence, exhibits a kind of lobe generation. If the main lobe is less than 1$\times$10$^{-6}$, the critical value can be safely considered before $x$ as shown in Figure (3a). In Figure (3b), the main lobe is higher than 1$\times$10$^{-6}$ so $R_c$ must be taken beyond it.

\[
E_c = \left| 1 - \frac{I_{\text{num}}(R)}{I_{\text{true}}(R)} \right|
\]  

(a) Weak Singularity 1/r

It has been found that when $R > 1$ then direct integration is more accurate due to the fact that the Jacobian of transformation will play undesired role. Also, when $1.0 < R < 0.75$, no significant improvement in accuracy is indicated for the same quadrature order, therefore direct method is preferable in that range. But when $R$ is less than 0.75, transformation method is found to be more efficient. It is clear from Table-1 which records the values of critical ratio for different quadratures that when 0.75 > $R$ > 0.08 the low degree transformation is more efficient than higher degree one. However, when $R$ is less than 0.08, the transform $T^i_f$ is found to be more efficient than $T$. By inspecting Table-1, the following formula can be concluded about the optimum value of $\beta$:

\[
\beta = 2 + K \ln \left( \frac{1}{R} \right)
\]

But a higher order quadrature is needed, order at least (1.5$\beta$) or (1.5$\beta$) in order to achieve the required accuracy, otherwise it will be degraded. It is clear that a quadrature less then 5$x$5 is not capable of handling transformation scheme and high error will arise if it is used. Increasing $\beta$ beyond 5 has an effect of reducing the critical ratio, but a higher order quadrature is needed. This is obvious when comparing...
$T^8$ and $T^3$ columns. Since $\beta$ is 6 for the former, a quadrature of order at least $\approx 81$ is needed to achieve better accuracy which is satisfied for all the quadratures higher than $8\times8$, but for other quadratures, the accuracy is affected. The proposed procedure to apply numerical integration is given as follows:

1. After evaluating $R$ from the previous section, for each sub-element, if $R > 0.5$ then using direct integration is preferable.
2. If $0.5 > R > 0.08$, using $T^2$ is preferable with the appropriate quadrature.
3. If $R < 0.08$ then use $T^3$ transform with quadrature of order at least $8\times8$.

The drawback in transformation method is the evaluation of transformed variables in the form of power functions which consumes a lot of CPU resources. To minimize this problem, repeated multiplication of the variable by itself can be used instead of power function. For that reason, $\beta$ must be kept integer as compromising between speed and accuracy. Thanks to real-time multiplication in math coprocessors, repetitive multiplication will consume very little time in most programming environments.

(b) Strong Singularity $1/\sqrt{r}$

The critical ratios for this type of singularity are listed in Table-2. In this case, using $T^2$ was found to be, in general, more accurate than transforms of higher degree when $R > 0.05$. However, if $R < 0.05$, higher degree transform has advantage over $T^2$, but a quadrature of order at least $(2\beta\times2\beta)$ or $(6\beta\times\beta)$ is needed to achieve the required accuracy. This is clear in Table-2 which lists the critical ratios for different integration schemes for the strong singularity. Note that $T^6$ has lower accuracy than $T^3$ when the order of quadrature is less than $12\times12$. The proposed procedure to apply numerical integration for strong singularity is:

1. If $(R > 0.5)$ then using direct integration is preferable.
2. If $(0.5 > R > 0.05)$, using $T^2$ is preferable with the appropriate quadrature.
3. If $(R < 0.05)$ then use $T^3$ or $T^6$ but a quadrature of order $10\times10$ is needed for the former and $12\times12$ for the latter scheme.

(c) Hyper Singularity $1/\sqrt{r^2}$

Here, $T^2$ transform is generally more accurate than other transforms when $R > 0.01$ as can be shown in Table-3. When $R < 0.01$, $T^2$ is found to be the most accurate transform with abrupt improvement in accuracy over $T^2$, but a quadrature of order at least $(3\beta\times3\beta)$ or $(9\beta\times\beta)$ is needed to achieve the required accuracy (in this case $16\times16$ or higher order is required). Note that $T^6$ is better than $T^6$ when the order of quadrature is less than $18\times18$ which confirms the previous conclusion. The proposed procedure to apply numerical integration is:

1. If $(R > 0.6)$ then using direct integration is preferable.
2. If $(0.5 > R > 0.01)$, using $T^3$ is preferable with the appropriate quadrature.
3. If \( R < 0.01 \) then use any one of the transforms according to the value of \( R \) but a quadrature of order at least \( 16 \times 16 \) is needed.

It can be shown that using high-order quadrature will significantly improve the results and reduces critical ratio. This is due to the fact that \( m \)-order Gauss quadrature can exactly integrate a polynomial of order up to \( (2m-1) \) [18].

6. Conclusions

So far, a powerful integration scheme has been introduced. This scheme involves the combination of two techniques, element subdivision and transformation of integration variables. The first technique reduces the critical ratio \( R \) and permits the application of the appropriate quadrature by comparing the evaluated ratio \( R \) with the critical ratios given in section 5. While the second technique weakens the singularity due to multiplication by the transformation Jacobian. Tables 1, 2 and 3 can be used to select the appropriate quadratures for various values of \( R \). For large values of the ratio \( R \), direct integration is more efficient, but when \( R \) is very small, transformation of variables is found to be superior over direct integration. A reduction in the critical ratio down to \( 1/500 \) has been obtained for the quadrature of order \( 18 \times 18 \) in hypersingular case. In fact, the efficiency is abruptly improved with the increase of quadrature order even for kernels of high order singularity. For intermediate values of \( R \), lower degree transformations are found to be more efficient than higher degree ones, but for very small values of \( R \), one must use higher degree transformations due to their improved efficiency.

The main disadvantage of the transformation method, resulting from the calculation of power functions, can be avoided by using repetitive multiplication instead of power functions. For that reason \( \beta \) is always chosen as an integer value. This is important to save CPU utility.

The proposed scheme can easily be extended to integrate over triangular-shaped elements calling that the inner period of integration can be expressed as a function of the outer abscissa [17]. Also, the proposed scheme can be used with super-singularities such as \( 1/R^4 \) and \( 1/R^6 \), but higher order Gauss quadratures are needed in order to achieve the desired accuracy.
Figure (3) Typical examples of relative error function $E_r$ vs. the ratio $R$

<table>
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<tr>
<th>Quadrature order</th>
<th>Critical Ratio $R_\alpha$</th>
<th>$T^1$</th>
<th>$T^2$</th>
<th>$T^3$</th>
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### Table 2: Critical ratios for strong singularity

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### Table 3: Critical ratios for hyper singularity

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References


