On A Maxi- Fuzzy subset of A Ring R

 حول المجموعة الضبابية Maxi على الحلقة R

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Abstract

In this research, we defined the term maxi-fuzzy subset of a ring R and studied some basic properties.

Also defined tow definitions of maxi-fuzzy integral domain of a ring R and we proof these are equivalents.

المستخلص:

قدمنا في هذا البحث مفهوم المجموعة الضبابية Maxi على الحلقة R ودرسنا بعض الخواص لتلك المجموعة فضلا عن ذلك قدمنا مفهوم المجموعة الضبابية Maxi التامة مع عبارة مكافئه له.

Introduction:

In 1965 Zadeh introduced the concepts about fuzzy set, in 1971 Rosenfeld formulated the term of fuzzy subgroup, in 1982 Liu formulated the term of fuzzy subring, in 2008 H.R.Yassein formulated the term of maxi-fuzzy subset of a group[2]. Our work aims to define the maxi-Fuzzy subset of a ring R and studied some properties.
1. Preliminary

Definition (1.1): Let X be a non-empty set. A fuzzy subset (set) of the X is a function \( \mu : X \rightarrow [0, 1] \). [1]

Definition (1.2): Let A be a non-empty set and \( x_t : A \rightarrow [0, 1] \) a fuzzy subset of A, where \( x \in A \) and \( t \in (0, 1] \) defined by:

\[
x_t(y) = \begin{cases} 
  t & \text{if } x = y \\
  0 & \text{if } x \neq y 
\end{cases}
\]

then \( x_t \) is called a fuzzy singleton. [4]

Definition (1.3): Let R be a ring \( \mu : R \rightarrow [0, 1] \), \( \mu \neq \emptyset \) then \( \mu \) is called a fuzzy subring of R if for every \( x, y \in R \)
1. \( \mu(x-y) \geq \min \{ \mu(x), \mu(y) \} \).
2. \( \mu(xy) \geq \min \{ \mu(x), \mu(y) \} \). [3]

Definition (1.4): A fuzzy subset \( \mu \) of a ring R is called a fuzzy ideal if for every \( x, y \in R \)
1. \( \mu(x-y) \geq \min \{ \mu(x), \mu(y) \} \).
2. \( \mu(xy) \geq \max \{ \mu(x), \mu(y) \} \).

Definition (1.5): Let \( \mu_1 \) and \( \mu_2 \) be two fuzzy subsets of a set \( X \) then the union of \( \mu_1 \) and \( \mu_2 \) denoted \( \mu_1 \cup \mu_2 \) is a fuzzy subsets of set \( X \) defined as

\( (\mu_1 \cup \mu_2)(x) = \max \{ \mu_1(x), \mu_2(x) \} \) for every \( x \in X \). [3]

Definition (1.6): Let \( \mu_1 \) and \( \mu_2 \) be two fuzzy subsets of a set \( X \) then the intersection of \( \mu_1 \) and \( \mu_2 \) denoted by \( \mu_1 \cap \mu_2 \) is a fuzzy subset \( S \) of \( X \) defined as

\( (\mu_1 \cap \mu_2)(x) = \min \{ \mu_1(x), \mu_2(x) \} \) for every \( x \in X \). [3]

2. Maxi-Fuzzy subset

Definition (2.1): Let R be a ring. A fuzzy subset of a ring R is called maxi-fuzzy subset of a ring R if for every \( x, y \in R \):
1. \( \mu(x-y) \geq \max \{ \mu(x), \mu(y) \} \).
2. \( \mu(xy) \geq \max \{ \mu(x), \mu(y) \} \).
Statement (2.2): Every maxi-fuzzy subset of ring is fuzzy subring.

The converse of this statement is not true, i.e., every fuzzy subring is not necessary maxi-fuzzy subset of ring. The following example explain this.

Example (2.3): Let Q be the ring of rational numbers \( \mu: Q \rightarrow [0, 1] \) define as follows:

\[
\mu(x) = \begin{cases} 
  1 & \text{if } x \in Z \\
  1/2 & \text{if } x \not\in Z
\end{cases}
\]

then \( \mu \) is a fuzzy subring of ring Q, but not maxi-fuzzy subset of ring Q.

If \( x \in Z, y \not\in Z \) then \( x-y \not\in Z \), therefore \( \mu(x-y) = 1/2, \) max \{\( \mu(x), \mu(y) \)\} = 1.

Hence \( \mu(x-y) < \max \{\mu(x), \mu(y)\} \).

Statement (2.4): Every maxi-fuzzy subset of ring is fuzzy ideal.

The converse of this statement is not true, i.e., every fuzzy ideal is not necessary maxi-fuzzy subset of ring. The following example explain this.

Example (2.5): Let \((Z_4,+_4,\cdot_4)\) be the ring of integer numbers modulo 4, \( \mu: Z_4 \rightarrow [0, 1] \) define as follows:

\[
\mu(x) = \begin{cases} 
  1/2 & \text{if } x = 0,2 \\
  1/3 & \text{if } x = 1,3
\end{cases}
\]

then \( \mu \) is a fuzzy ideal of ring \( Z_4 \), but not maxi-fuzzy subset of ring \( Z_4 \).

( \( \mu(3-2)= \mu(1)= 1/3, \) max \{\( \mu(3), \mu(2) \)\}=1/2 )

Hence \( \mu(x-y) < \max \{\mu(x), \mu(y)\} \).

Proposition (2.6): Let \( \mu_1 \) and \( \mu_2 \) be two maxi-fuzzy subset of ring \( R \) then \( \mu_1 \cap \mu_2 \) is a maxi-fuzzy subset of ring \( R \).

Proof:

1) \( (\mu_1 \cap \mu_2)(x-y) = \min \{\mu_1(x-y), \mu_2(x-y)\} \)

\[ \geq \min \{\max \{\mu_1(x), \mu_1(y)\}, \max \{\mu_2(x), \mu_2(y)\}\} \]

\[ \geq \max \{\min \{\mu_1(x), \mu_2(x)\}, \min \{\mu_1(y), \mu_2(y)\}\} \]

\[ = \max \{((\mu_1 \cap \mu_2)(x)), ((\mu_1 \cap \mu_2)(y))\} \]

2) \( (\mu_1 \cap \mu_2)(xy) = \min \{\mu_1(xy), \mu_2(xy)\} \)

\[ \geq \min \{\max \{\mu_1(x), \mu_1(y)\}, \max \{\mu_2(x), \mu_2(y)\}\} \]
\[ \geq \max \{ \min \{ \mu_1(x), \mu_2(x) \}, \min \{ \mu_1(y), \mu_2(y) \} \} \]
\[ = \max \{ (\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y) \} \]

Hence \( \mu_1 \cap \mu_2 \) is a maxi-fuzzy subset of the ring \( R \). □

Proposition (2.7): Let \( \mu_1 \) and \( \mu_2 \) be two maxi-fuzzy subsets of ring \( R \) then \( \mu_1 \cup \mu_2 \) is a maxi-fuzzy subset of ring \( R \).

Proof:
1) \( (\mu_1 \cup \mu_2)(x-y) = \max \{ \mu_1(x-y), \mu_2(x-y) \} \)
\[ \geq \max \{ \max \{ \mu_1(x), \mu_1(y) \}, \max \{ \mu_2(x), \mu_2(y) \} \} \]
\[ = \max \{ \max \{ \mu_1(x), \mu_2(x), \mu_1(y), \mu_2(y) \} \} \]
\[ = \max \{ \max \{ \mu_1(x), \mu_2(x) \}, \max \{ \mu_1(y), \mu_2(y) \} \} \]
\[ = \max \{ (\mu_1 \cup \mu_2)(x), (\mu_1 \cup \mu_2)(y) \} \]

2) \( (\mu_1 \cup \mu_2)(xy) = \max \{ \mu_1(xy), \mu_2(xy) \} \)
\[ \geq \max \{ \max \{ \mu_1(x), \mu_1(y) \}, \max \{ \mu_2(x), \mu_2(y) \} \} \]
\[ = \max \{ \max \{ \mu_1(x), \mu_2(x), \mu_1(y), \mu_2(y) \} \} \]
\[ = \max \{ \max \{ \mu_1(x), \mu_2(x) \}, \max \{ \mu_1(y), \mu_2(y) \} \} \]
\[ = \max \{ (\mu_1 \cup \mu_2)(x), (\mu_1 \cup \mu_2)(y) \} \]

Hence \( \mu_1 \cup \mu_2 \) is a maxi-fuzzy subset of ring \( R \). □

Definition (2.8): Let \( \mu \) be a maxi-fuzzy subset of \( R \) then \( \mu \) is called an maxi-fuzzy integral domain if \( xy=0 \), and \( \max \{ \mu(x), \mu(y) \} > 0 \) then \( x=0 \) or \( y=0 \).

Definition (2.9): Let \( \mu \) be a maxi-fuzzy subset of ring \( R \) then singleton \( a_t \subseteq \mu \) with \( t \in (0,1] \), \( a_t \) be non-zero fuzzy singleton \( (a_t \neq 0) \) is said to be zero divisor fuzzy singleton if there exists \( b_s \subseteq \mu \) with \( t \in (0,1] \), \( b_s \) is non-zero fuzzy singleton such that \( a_t b_s = 0_\lambda \) and \( b_s a_t = 0_\lambda \) when \( \lambda = \min \{s,t\} \).

Definition (2.10): A maxi-fuzzy subset \( \mu \) of ring \( R \) is said to be an maxi-fuzzy integral domain if has no zero divisor fuzzy singleton \( ( \text{if } x_t \subseteq \mu, \exists y_t \subseteq \mu \text{ such that } x_t, y_t = 0_t \text{ with } t \in (0,1] \implies x_t = 0_t \text{ or } y_t = 0_t ) \).

Theorem (2.11): Let \( \mu \) be a maxi-fuzzy subset of ring \( R \). Then \( \mu \) is an maxi-fuzzy integral domain if and only if \( \mu \) has no zero divisor fuzzy singleton.
Proof:

1) Suppose $\mu$ is an maxi- fuzzy integral domain holds and prove $\mu$ has no zero divisor fuzzy singleton.

If $x_t \cdot y_t = 0_t$, we must prove $x_t = 0_t$ or $y_t = 0_t$ with $t \in (0,1]$.

$x_t \cdot y_t = 0_t$ implies $\forall z \in \mathbb{R}, (xy)_t(z) = 0_t(z)$

then

$$(xy)_t(z) = \begin{cases} t & \text{if } xy = z \\ 0 & \text{if } xy \neq z \end{cases} = \begin{cases} t & \text{if } 0 = z \\ 0 & \text{if } 0 \neq z \end{cases} = 0_t(z)$$

Hence

$$(xy)_t(z) = \begin{cases} t & \text{if } xy = z \\ 0 & \text{if } xy \neq z \end{cases} = \begin{cases} t & \text{if } 0 = z \\ 0 & \text{if } 0 \neq z \end{cases} = \begin{cases} t & \text{if } xy = z = 0 \\ 0 & \text{if } xy \neq z, z \neq 0 \end{cases}$$

by the definition (2.7) implies $x=0$ or $y=0$

$$(xy)_t(z) = \begin{cases} t & \text{if } x = 0 \text{ or } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now if $x=0$

$$(xy)_t(z) = \begin{cases} t & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} = x_t(0) = 0_t(x)$$

Suppose $x_t = u = 0_t$ then $x_t(u) = 0_t(u)$

Hence $x_t = 0_t$ with $t \in (0,1]$.

Similarly, we can prove $y_t = 0_t$ with $t \in (0,1]$ if $y=0$.

2) Suppose $\mu$ has no zero divisor fuzzy singleton holds and prove $\mu$ is an maxi- fuzzy integral domain.

if $xy=0$ and $\max\{\mu(x), \mu(y)\} > 0$, we must prove $x=0$ or $y=0$.

Now, if $xy=0$ then $(xy)_t=0_t$ with $t \in (0,1)$

then by definition (2.9) $x_t = 0_t$ or $y_t = 0_t$.

if $\forall u,v \in \mathbb{R}, x_t(u) = 0_t(u)$ or $y_t(v) = 0_t(v)$

then

$$\begin{cases} t & \text{if } x = u \\ 0 & \text{if } x \neq u \end{cases} = \begin{cases} t & \text{if } 0 = u \\ 0 & \text{if } 0 \neq u \end{cases}$$
or

\[
\begin{cases}
    t & \text{if } y = v \\
    0 & \text{if } y \neq v
\end{cases}
= \begin{cases}
    t & \text{if } 0 = v \\
    0 & \text{if } 0 \neq v
\end{cases}
\]

\[x_i(u) = \theta_i(u) = \begin{cases}
    t & \text{if } x = u = 0 \\
    0 & \text{if } x \neq u, 0 \neq v
\end{cases}\]

\[y_i(v) = \theta_i(v) = \begin{cases}
    t & \text{if } y = v = 0 \\
    0 & \text{if } y \neq v, 0 \neq v
\end{cases}\]

Hence \( x = 0 \) or \( y = 0 \)

References

1- D. Dubois, (1980), Fuzzy Sets and Systems, University Paul Sobatier, New York.
3- W. B. Vasantha, (2003), Smarandache Fuzzy Algebra, Indian Institute of Technology Madras.