Investigation of solvability condition for sixth-order boundary value problem

Akram Hassan Mahmood
Department of Mathematics
College of Education
University of Mosul

Alaa Ahmed Mohamed
M.SC. of Mathematics

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Abstract
This paper is concerned with the solvability condition for nonhomogenous linear boundary value problem for sixth-order ordinary differential equation.

Throughout this study, we observed that, when the homogenous problem have nontrivial solution, then the nonhomogenous boundary value problem have a solution in case of nonhomogenous term that satisfied the solvability condition.

We justified our results through the given example.

Keywords: Sixth-order boundary value problem, self-adjoint problem.

(1) - Introduction
There is a relationship between homogenous and nonhomogenous linear boundary value problem as there is between homogenous and nonhomogenous linear algebra system. A nonhomogenous boundary value problem has a unique solution and the corresponding homogenous problem has only the trivial solution, then a nonhomogenous problem has either no solution or infinity many, and the corresponding homogenous

This paper deals with investigation of the solvability for the following boundary value problem
\[ p_6(x)\phi^{(6)} + p_5(x)\phi^{(5)} + p_4(x)\phi^{(4)} + p_3(x)\phi^{(3)} + p_2(x)\phi^{(2)} + p_1(x)\phi' + p_0(x)\phi = f(x) \quad \ldots(1.1) \]
\[ \phi(a) = \beta_1, \phi'(a) = \beta_2, \phi''(a) = \beta_3, \phi(b) = \beta_4, \phi'(b) = \beta_5, \phi''(b) = \beta_6 \quad \ldots(1.2) \]

Where \( P_i(x) \in c^i[a,b], i=1,2,3,4,5,6, \) \( P_6(x) \neq 0 \) in the interval \([a,b]\) and \( f(x) \) is continuous function in the same interval and also \( \beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \text{ and } \beta_6 \) are real constants.

**2)- Solvability Condition for the problem**

In this section, we will try to give a theorem which is the main basis of solvability condition for the problem (1.1). It is worth noting that when the homogenous have a nontrivial solution, the nonhomogenous equations have a solution if and only if the nonhomogenous parts satisfy a solvability condition [4].

**Theorem**

The desired solvability condition that the problem (1.1-1.2) has a solution is
\[ [P_6u^{(6)} + (4P_6u^{(5)} + P_6u^{(4)} - P_5u^{(3)})\beta_3 - (10P_6u^{(4)} + 5P_6u^{(3)} + P_6u^{(2)})\beta_2 - 4P_5u^{(5)} - P_5u^{(4)} + P_4u^{(3)})\beta_4] - [P_6u^{(6)} + (4P_6u^{(5)} + P_6u^{(4)} - P_5u^{(3)})\beta_3 - (10P_6u^{(4)} + 5P_6u^{(3)} + P_6u^{(2)})\beta_2 - 4P_5u^{(5)} - P_5u^{(4)} + P_4u^{(3)})\beta_4] = \int_a^bf(x)u(x)dx \quad \ldots(2.1) \]

**Proof**

To determine the solvability condition for the problem (1.1-1.2) we multiply (1.1) by \( u(x) \) and integrate it from \( x = a \) to \( x = b \), we obtain
\[ \int_a^bP_6u\phi^{(6)}dx + \int_a^bP_5u\phi^{(5)}dx + \int_a^bP_4u\phi^{(4)}dx + \int_a^bP_3u\phi^{(3)}dx + \int_a^bP_2u\phi^{(2)}dx + \int_a^bP_1u\phi' dx + \int_a^bP_0u\phi = \int_a^bf(x)u(x)dx \quad \ldots(2.2) \]
We integrate by parts the integrals in (2.2) to transfer the derivatives from $\phi$ to $u$, we note that
\[
\int_{a}^{b} P_{6} u \phi' \; dx = \left( P_{6} u \phi \right)^{b}_{a} - \left( P_{6} u \phi' \right)^{b}_{a} = \left[ P_{6} u \phi - \left( P_{6} u \right)' \phi'^{b}_{a} \right] - \int_{a}^{b} (P_{6} u)^{''} \phi' \; dx = \left[ P_{6} u \phi - \left( P_{6} u \right)' \phi'^{b}_{a} \right] + \left( P_{6} u \right)'' \phi'^{b}_{a} \]
\[
- \int_{a}^{b} (P_{6} u)^{''} \phi'^{b}_{a} = \left[ P_{6} u \phi - \left( P_{6} u \right)' \phi'^{b}_{a} \right] + \left( P_{6} u \right)'' \phi'^{b}_{a} \]
\[
- \int_{a}^{b} (P_{6} u)^{''} \phi'^{b}_{a} = \left[ P_{6} u \phi - \left( P_{6} u \right)' \phi'^{b}_{a} \right] + \left( P_{6} u \right)'' \phi'^{b}_{a} \]
In similar way
\[
\int_{a}^{b} P_{3} u \phi'^{b}_{a} = \left[ P_{3} u \phi'^{b}_{a} \right] \quad \int_{a}^{b} P_{4} u \phi'^{b}_{a} = \left[ P_{4} u \phi'^{b}_{a} \right] \quad \int_{a}^{b} P_{5} u \phi'^{b}_{a} = \left[ P_{5} u \phi'^{b}_{a} \right] \quad \int_{a}^{b} P_{6} u \phi'^{b}_{a} = \left[ P_{6} u \phi'^{b}_{a} \right]
\]
Therefore we can rewrite (2.2) as
\[
\int_{a}^{b} \left[ \phi \left( P_{6} u \right)'' - (P_{3} u)'' + (P_{4} u)'' - (P_{3} u)' \right] \phi^{dx},
\]
\[
+ \left( P_{6} u \phi'^{b}_{a} \right) - \left( P_{6} u \right)' \phi'^{b}_{a} = \left( P_{6} u \right)'' \phi'^{b}_{a} + \left( P_{6} u \right)' \phi'^{b}_{a} \]
\[
- \left( P_{6} u \right)'' \phi'^{b}_{a} + \left( P_{6} u \right)' \phi'^{b}_{a} = \left( P_{6} u \right)'' \phi'^{b}_{a} + \left( P_{6} u \right)' \phi'^{b}_{a} \]
\[
- \left( P_{6} u \right)'' \phi'^{b}_{a} + \left( P_{6} u \right)' \phi'^{b}_{a} = \left( P_{6} u \right)'' \phi'^{b}_{a} + \left( P_{6} u \right)' \phi'^{b}_{a} \]
To find the differential equation describing the adjoint $u$, we set the coefficient of $\phi$ in the integral on the left-hand side of (2.3) equal zero, we have:
\((P_2u)^{VI} - (P_3u)^{V} + (P_4u)^{IV} -(P_5u)^{IV} + (P_6u)^{IV} - (P_7u)^{IV} + (P_8u) = 0 \) \hspace{1cm} (2.4)

which is the adjoint homogenous differential equation corresponding to (1.1). In order that the homogenous differential equation (1.1) be self - adjoint, (2.4) must be the same as the homogenous equation (1.1).

Expanding the derivatives in (2.4) and obtain

\[
P_6u^{VI} + (6P_6' - P_5)u^{V} + (15P_6'' - 5P_5' + P_4)u^{IV} + (20P_6''' - 10P_5'' + 4P_4')
\]

\[
- P_3'u''' + (15P_6^{IV} - 10P_5'' + 6P_4'' - 3P_3' + P_2')u'' + (6P_6^{IV} - 5P_5^{IV} + 4P_4'')
\]

\[
- 3P_3'' + 2P_4' - P_1'u' + (P_6^{IV} - P_5^{IV} + P_4^{IV} - P_3'' + P_2'' - P_1' + P_0)u = 0 \] \hspace{1cm} (2.5)

Comparing (2.5) with (1.1) we obtain

\[
P_5 = 6P_6' - P_5
\]

\[
P_4 = 15P_6'' - 5P_5' + P_4
\]

\[
P_3 = 20P_6''' - 10P_5'' + 4P_4' - P_3
\]

\[
P_2 = 15P_6^{IV} - 10P_5'' + 6P_4'' - 3P_3' + P_2
\]

\[
P_1 = 6P_6^{V} - 5P_5^{IV} + 4P_4'' - 3P_3'' + 2P_4' - P_1
\]

\[
P_0 = P_6^{VI} - P_5^{V} + P_4^{IV} - P_3'' + P_2'' - P_1' + P_0
\]

or

\[
P_5 = 3P_6' \hspace{1cm}, \hspace{1cm} P_3 = -5P_6'' + 2P_4' \hspace{1cm}, \hspace{1cm} P_1 = 3P_6^{V} - P_4'' + P_2'
\]

Then (1.1) becomes

\[
P_6\phi^{VI} + 3P_6'\phi^{V} + P_4\phi^{IV} + (2P_4'u - 5P_6''\phi'^{IV} + P_2\phi'' + (3P_6' - P_4'' + P_2')\phi' + P_0\phi = 0
\]

Which can be written as

\[
\frac{d^3}{dx^3}(P_6\phi^{IV}) + \frac{d^2}{dx^2}((P_4 - 3P_6'')\phi'^{IV}) + \frac{d}{dx}((P_2 - P_4'' + 3P_6^{IV})\phi') + P_0\phi = 0 \] \hspace{1cm} (2.6)

To determine the boundary conditions for \(u\), we consider the homogenous problem that is: put \(f = 0\) in (2.3) and using (2.4), we have

\[
\left\{(P_6u)^{V} - (P_7u)^{V} - (P_3u)^{IV} + (P_4u)^{IV} - (P_5u)^{IV} - (P_6u)^{IV}
\right\}
\]

\[
- (P_3u)^{IV} + (P_4u)^{IV} - (P_5u)^{IV} + (P_6u)^{IV} - (P_7u)^{IV} - (P_8u) = 0 \] \hspace{1cm} (2.7)

But for the homogenous problem

\[
\phi(a) = \phi'(a) = \phi''(a) = \phi(b) = \phi'(b) = \phi''(b) = 0
\]

Hence (2.7) becomes

\[
P_6u_{|b} \phi''(b) - \left[(P_6u) - (P_7u)\right]_{b} \phi''(b) + \left[(P_6u)'' - (P_3u)' + (P_4u)\right]_{b} \phi''(b)
\]

\[
- P_6u_{|a} \phi'(a) + \left[(P_6u) - (P_7u)\right]_{a} \phi'(a) - \left[(P_6u)'' - (P_3u)' + (P_4u)\right] \phi''(a) = 0 \] \hspace{1cm} (2.8)

We choose the adjoint boundary conditions such that each of the coefficients of
(3)- Illustrative Example

This example concerns the solvability condition of the following boundary problem:

\[ y'' + 56\pi^2 y'' + 784\pi^4 y'' + 2304\pi^6 y = \pi^5 \cos 2\pi x \]

\[ y(0) = \beta_1 , y'(0) = \beta_2, y''(0) = \beta_3, y(1) = \beta_4, y'(1) = \beta_5, y''(1) = \beta_6 \]

Using the theorem we have

\[ p_6 = 1, p_5 = 0, p_4 = 56\pi^2, p_3 = 0, p_2 = 784\pi^4, p_1 = 0, p_0 = 2304\pi^6 \]

The general solution of boundary value problem is

\[ y = c_1 \cos 2\pi x + c_2 \sin 2\pi x + c_3 \cos 4\pi x + c_4 \sin 4\pi x + c_5 \cos 6\pi x + \]

\[ + \ c_6 \sin 6\pi x + \frac{1}{1536} x \sin 2\pi x \]

The general solution of adjoint equation is

\[ u = \frac{5}{3} \cos 2\pi x - 5 \sin 2\pi x - \frac{8}{3} \cos 4\pi x + \sin 4\pi x + \cos 6\pi x + \sin 6\pi x \]

and the adjoint boundary condition

\[ u(0) = u'(0) = u''(0) = 0, u'''(0) = -240\pi^3, u''(0) = 640\pi^4, u'(0) = 8640\pi^5 \]

\[ u(1) = u'(1) = u''(1) = 0, u'''(1) = -240\pi^3, u''(1) = 640\pi^4, u'(1) = 8640\pi^5 \]

Hence the solvability condition

\[ -240\pi^3 \beta_6 + 640\beta_5 - (8640\pi^5 + 56\pi^2(-240)) \beta_4 - (-240\pi^3 \beta_3 + 640\pi^4 \beta_2 -
\]

\[ -(8640\pi^5 + 56\pi^2(-240)) \beta_1 \]

\[ \Rightarrow -240\pi^3 (\beta_6 - \beta_3) + 640\pi^4 (\beta_5 - \beta_2) + 4800\pi^5 (\beta_4 - \beta_1) = \frac{5}{6} \pi^5 \]
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If \( \beta_6 = \beta_3, \beta_4 = \beta_1, \beta_5 - \beta_2 = \frac{2\pi}{1536} \)

We have \( \frac{5}{6}\pi = \frac{5}{6}\pi \)

That is \( \text{R.H.S} = \text{L.H.S} \)

(4)- Conclusions

Throughout the investigation of the solvability condition of sixth-order boundary value problem (1.1-1.2), we have seen that if the homogenous problem has non trivial solution, the corresponding nonhomogenous problem has a solution if the nonhomogenous term satisfied the condition (2.1).

References


