ON THE MEASURABLE FAMILIES OF Š- BANACH LATTICE

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ABSTRACT:
The aim of this paper is to study a measurable families of \( Š \)- Banach lattice and their decomposition of a separable Banach Š- vector lattice into measurable field of Banach lattice. Moreover there exists a Freudenthal unit in \( \bar{X} \) if and only if it exists in \( X_\tau \).

1. Introduction:
In this paper, we introduce the important definitions and some information about the measurable field of metric space and of Boolean algebras that we needed it in this work. We get the main results about the decomposition of separable Banach Š- vector lattice into a measurable field of Banach lattice.

2. The preliminaries
We recall the basic definitions and information which are needed in our work.

2.1 Definition: [11]
A mapping \( \rho : X \times X \rightarrow \hat{S} \) is called a metric on a set \( X \) with values in \( \hat{S} \) if

1. \( \rho(x,y) \geq 0 \) for any \( x,y \in X \) and \( \rho(x,y) = 0 \) if and only if \( x=y \).
2. \( \rho(x,y) = \rho(y,x) \) for any \( x,y \in X \).
3. \( \rho(x,y) \leq \rho(x,z) + \rho(z,y) \) for any \( x,y,z \in X \).

2.2 Remark: [8]
Suppose that \( T \) is the interval \([0,1]\) , let \( A \) be the \( \sigma \)- algebra of the Lebesgue measurable subset of \( T \), and \( P \) be the Lebesgue measure on \( T \).
We shall denote by \( \hat{S} \) the ring of all real measurable functions on \([0,1] \).
Definition: [11]

The space \((X, \rho)\) is called separable, if there exists a countable subset \(M \subseteq X\) such that for any \(x \in X\), there is \(\{z_n\}_{n=1}^{\infty} \subseteq M\) for which \(\rho(x, z_n) \xrightarrow{(t)} 0\).

Suppose that \((X, \rho_T)\) be a complete separable metric space defined for \(P\)-almost every \(\tau \in T\).

Definition: [11]

A measurable field of metric space is a pair \(\{X, \{X_\tau\}_{\tau \in T}\}\) where \(X\) is a totality of functions \(x: \tau \to x(\tau) \in X_\tau\) for \(P\)-almost every \(\tau \in T\) such that:

1. There exists a sequence \(\{x_n\}_{n=1}^{\infty} \subseteq X\) such that \(\{x_n(\tau)\}_{n=1}^{\infty} \) is dense in \((X_\tau, \rho_\tau)\) for \(P\)-almost every \(\tau \in T\).
2. The function \(\tau \to \rho_\tau(x(\tau), y(\tau))\) is measurable on \((T, \mathcal{A}, P)\) for all \(x, y \in X\).
3. If \(\{y_n\}_{n=1}^{\infty} \subseteq X, y : \tau \to y(\tau) \in X_\tau\) for \(P\)-almost every \(\tau \in T\) and \(\rho_\tau(y_n(\tau), y(\tau)) \to 0\) as \(n \to \infty\) for \(P\)-almost every \(\tau \in T\), then \(y \in X\).

Remark:[9,10]

Let \(\{X, \{X_\tau\}_{\tau \in T}\}\) be a measurable field of metric space and let \(\hat{X}\) be the set of all classes for \(P\)-almost everywhere coinciding elements from \(X\).

For any \(\hat{x}, \hat{y} \in \hat{X}\), we denote by \(\hat{\rho}(\hat{x}, \hat{y})\) the class from \(\hat{S}\) containing the function \(\rho_\tau(x(\tau), y(\tau))\), where \(x(\tau)\) and \(y(\tau)\) are representatives of the classes \(\hat{x}\) and \(\hat{y}\), respectively. Then \((\hat{X}, \hat{\rho})\) is a complete separable space with a \(\hat{S}\) - valued metric.

Definition: [11]

A measurable field of metric space is said to be saturated if the following two conditions are satisfied:

1. \(y: \tau \to y(\tau) \in X_\tau\) for \(P\)-almost every \(\tau \in T\).
2. The function \(\tau \to \rho_\tau(x(\tau), y(\tau))\) is measurable on \((T, \mathcal{A}, P)\) for any \(x \in X\), imply \(y \in X\).

Remark: [11]

Let \(\{X, \{X_\tau\}_{\tau \in T}\}\) be a universal metric space of Uryson (one can consider that \(U=\mathbb{C}[0,1]\) with the uniform metric.

In [9] it is shown that any complete separable space \(X\) with an \(\hat{S}\)-valued metric is \(\hat{S}\)-isometric to some closed subset of \(\hat{S}(T, Z)\).

Now, let \(\{X, \{X_\tau\}_{\tau \in T}\}\) be a measurable field of metric space, and let \(A_\tau\) be closed subset of \(X_\tau\) for \(P\)-almost every \(\tau \in T\).
2.9 Definition: [9]

A measurable field of closed sets is a pair \( \{ A, \{ A_\tau \}_{\tau \in \mathcal{T}} \} \) where \( A \subseteq X \) and

1. If \( x \in A \), then \( x(\tau) \in A_\tau \) for \( P \)-almost every \( \tau \in \mathcal{T} \).
2. There exists \( \{ x_n \}_{n=1}^\infty \subset A \) such that \( \{ x_n(\tau) \}_{n=1}^\infty \) is dense in \( A_\tau \) for \( P \)-almost every \( \tau \in \mathcal{T} \).
3. If \( \{ y_n \}_{n=1}^\infty \subset A, y \in X \) and \( \rho(\tau, y) \rightarrow 0 \) as \( n \rightarrow \infty \) for \( P \)-almost every \( \tau \in \mathcal{T} \), then \( y \in A \).

2.10 Remark:[10]

We say that \( X \) be a bimodule over \( \hat{S} = [0,1] \), i.e. \( X \) is abelian group with respect to addition operation (+) and right and left multiplication by element from \( \hat{S} \) are defined on \( X \) having the properties:

1. \( \lambda(x+y) = \lambda x + \lambda y \), \( (x+y)\lambda = x\lambda + y\lambda \)
2. \( (\lambda + \mu)x = \lambda x + \mu x \), \( x(\lambda + \mu) = x\lambda + x\mu \)
3. \( \lambda(\mu x) = (\lambda \mu)x \), \( (\lambda \mu)\mu = x(\lambda \mu) \)
4. \( \lambda \cdot x = x \cdot \lambda \) for all \( x, y \in X, \lambda, \mu \in \hat{S} \)

2.11 Definition: [2, 4, 7]

The collection \( B \) of Borel sets of a topological space \( X \) is the smallest \( \sigma \)-algebra containing all open sets of \( X \). That is, in addition to containing open sets, \( B \) must be closed under complement and countable intersections (and, thus, is also closed under countable unions).

2.12 Definition: [12, 13]

A Borel mapping is a mapping such that the inverse image of every Borel set is Borel. It is called Borel function.

2.13 Definition:[3]

A set \( C \subseteq \mathbb{R}^n \) is called convex if for all \( x, y \in C \), we have \( \lambda x + (1-\lambda)y \in C \) for all \( \lambda \in [0, 1] \).

2.14 Definition:[5]

A Banach space is a normed linear space in which every Cauchy sequence is convergent.

We cite now example of \( S^* \)-vector lattices

2.15 Example:[10]

Let \( P \) be the Lebesgue measure on \( \mathcal{V}(S^*) \), \( L_1(\mathcal{V}(S^*), P) \) be the Banach space of all \( P \)-integrable functions on \( ([0,1], P) \), \( \mathcal{V}_1 \) be a \( \sigma \)-subalgebra of \( \mathcal{V}(S^*) \) isomorphic to \( \mathcal{V}(S^*) \), \( E: L_1(\mathcal{V}(S^*), P) \rightarrow L_1(\mathcal{V}_1, m) \) be the conditional expectation.

Then \( \mu(e) = E(e) \) is a strictly positive measure on \( \mathcal{V}(S^*) \) with values in the \( \sigma \)-complete sublattice \( S_1 \) of all functions from \( S^* \) which are measurable with respect to \( \mathcal{V}_1 \) (it is clear that \( S_1 \) can be identified with \( S^* \), since \( \mathcal{V}_1 \) is isomorphic to \( \mathcal{V}(S^*) \).

In addition, \( \mu(eg) = e\mu(g) \) for any \( e \in \mathcal{V}_1, g \in \mathcal{V}(S^*) \).
It is clear that $S^*$ is a normal $S_1$-module.

Denote by $\hat{\mu}$ the integral constructed with respect to the measure $\mu$. For every number $P \geq 1$ we set

$$L_p(\mathcal{V}(S^*), \mu) = \{ x \in S^* : \hat{\mu}(|x|^p) \in S \}$$

And

$$\|x\|_p = (\hat{\mu}[|x|^p])^{1/p}$$

It is shown in [3,4] that $L_p(\mathcal{V}(S^*), \mu, \|\cdot\|_p)$ is a Banach $S_1$-vector lattice.

### 3. The Main Results:

This section is devoted to the main results concerning the measurable families of Banach lattice and decomposition of separable Banach $\hat{S}$-vector lattice of ordinary Banach lattices.

We begin with the useful information that used in our paper.

### 3.1 Definition: [10]

A measurable field $X$ of Banach spaces $(X_\tau, \|\cdot\|_\tau)$, $\tau \in T = ([0,1])$, $P$ is a set of functions $x: \tau \to x(\tau) \in X$, defined $p$-almost everywhere on $T$ and such that

1. The function $\tau \to \|x_\tau\|_\tau$ is measurable on $T$ for all $x \in X$;
2. If $x, y \in X$, $\alpha, \beta \in \hat{S}$, then the function $\alpha \cdot x(\tau) + \beta \cdot y(\tau)$ also belongs to $X$.
3. There exists a sequence $\{x_n\} \subset X$ such that the set $\{x_n(\tau)\}$ is dense in $X_\tau$ for $p$-almost every $\tau \in T$.
4. If $\{y_n\} \subset X$ and $y: \tau \to y(\tau) \in X_\tau$ is a function and $\|y_n(\tau) - y(\tau)\|_\tau \to 0$ as $n \to \infty$ for $p$-almost every $\tau \in T$, then $y \in X$.

It is clear that a measurable field Banach space $X$ is also a measurable field metric space $(X_\tau, d_\tau)$, $\tau \in T$, where

$$d_\tau(x(\tau), y(\tau)) = \|x(\tau) - y(\tau)\|_\tau, \ x(\tau), y(\tau) \in X_\tau.$$ 

### 3.2 Note: [10]

A measurable field Banach space $(X_\tau, \|\cdot\|_\tau)$ with property (1,2) of definition (2.1) is said to be saturated.

In the sequel the record $\{X_\tau, \tau \in T\}$ will mean that $X$ is a measurable field Banach space $(X_\tau, \|\cdot\|_\tau)$.

Now, for any measurable field Banach space $X$ denoted by $\hat{X}$ the set of all classes $\hat{x}$ of $p$-almost everywhere equal functions $x: \tau \to x(\tau)$ from $X$.
The pointwise operations:

\[ x + y : \tau \rightarrow x(\tau) + y(\tau), \]
\[ ax : \tau \rightarrow a(\tau)x(\tau), \]
\[ xa : \tau \rightarrow x(\tau)a(\tau), \]

\( x, y \in X, a \in \hat{S} \) and the function \( \| . \| : \tau \rightarrow \| x(\tau) \|_\tau, x \in X \), determine on \( \hat{X} \) a structure of a bimodule over \( \hat{S} \) with the \( \hat{S} \) – valued norm \( \| . \| _{\hat{X}} \). In addition, \((\hat{X}, \| . \| _{\hat{X}})\) is a separable bimodule, i.e. there exists a countable set \( \{ \hat{x}_n \} \subset \hat{X} \) such that, for any

\( \hat{x} \in \hat{X} \) there exists a subsequence \( \{ \hat{x}_{n_k} \} \) for which \( \| \hat{x} - \hat{x}_{n_k} \| _{\hat{X}}^{(o)} \rightarrow 0 \).

3.3 Remark:[12]

Suppose \( \hat{X} \) be a separable Banach \( \hat{S} \) – module with partial order in which represents a lattice coincide with the norm in \( \hat{X} \).

From \( |\hat{x}| \leq |\hat{y}| \) imply that \( \| \hat{x} \| \leq \| \hat{y} \| \) for each \( \hat{x}, \hat{y} \in \hat{X} \).

3.4 Definition:[6]

A vector lattice \( X \) is **Archimedean** whenever the relation \( 0 \leq nx \leq y, n \in N \), imply that \( x = 0 \).

3.5 Proposition:

Let \( \hat{X} \) be the set of all classes for \( P \)- almost everywhere coinciding elements from \( X \) ,The Archimedean condition is satisfied in \( \hat{X} \).

**Proof**:

Let \( \hat{x} \geq 0 \) and \( n\hat{x} \leq \hat{y} \), \( n = \frac{1}{1, \infty} \)

Then\( \| n\hat{x} \| \leq \| \hat{y} \| \), or \( \| \hat{x} \| \leq \frac{1}{n} \| \hat{y} \| \rightarrow 0 \) as \( n \rightarrow \infty \)

i.e.\( \hat{x} = 0 \)

Therefore

\[ |x \lor z - y \lor z| \leq |x - y| \quad \ldots \quad (1) \]
3.6 Definition:[14]

Let $X$ be a vector space over the real field $\mathbb{R}$. A nonempty convex subset $P$ of $X$ is called a cone if $\lambda P \subseteq P$ for all $\lambda \geq 0$.

3.7 Theorem:

Let $\{X,\{X_t\}_{t \in T}\}$ be a measurable field Banach space generating $\hat{X}$, then for almost everywhere $t \in T$ we can define a Banach Lattice on $X_t$ such that, the order in $X_t$ for almost everywhere $t \in T$ induces the order in $\hat{X}$.

Proof:

Let $\hat{G}$ be a countable dense subset of $\hat{R}$ such that:

1. $\hat{G} + \hat{G} \leq \hat{G}$.
2. $r \hat{G} \leq \hat{G}$ for all rational numbers $r \geq 0$.
3. $\hat{G} \cap (-\hat{G}) = 0$.

Now, suppose that $\{G_t\}_{t \in T}$ generated $\hat{G}$, such that, for almost everywhere countable dense subset $G_t$ of $\hat{R}$.

Since $G_t$ and $G_t$ are countable for almost everywhere $t \in T$, then the properties (1 - 3) are satisfied in $G_t$ for almost everywhere $t \in T$.

Thus, $K_t$ is a cone for each $t \in T$.

In fact, we need to prove $K_t \subseteq K_t$ for each $t \in R$, but, that is clear when we use the property (2) and consider the fact that $K_t$ is closed.

Thus, for almost everywhere $t \in T$, we can define a partial order induced the same order in $\hat{X}$.

Now, we need to prove that $X_t$ is a Banach lattice for that order, i.e. there exist $x_t \vee y_t$ for any $x_t, y_t \in X_t$.

From $|x_t| \leq |y_t|$ we have $\|x_t\|_t \leq \|y_t\|_t$.

Let $\tilde{F}$ be a countably densely subset of $\hat{X}$ such that $\hat{F} \cap \tilde{F} \subseteq \hat{F}$ and let $\{F_t\}_{t \in T}$ be a family of a countable dense set in $X_t$, for almost everywhere $t \in T$ generating $\tilde{F}$.

Therefore, almost everywhere $t \in T$ and for all $x_t, y_t \in \Gamma_t$, we can define $x_t \vee y_t$ such that $\Gamma_t \vee \Gamma_t \subseteq \Gamma_t$ and therefore the relation (1) and the order conditions are satisfied on $\Gamma_t$.

Let $y_t \in \Gamma_t, x_t \in X_t$ and $\{x^T_n\}_{n=1}^{\infty} \subseteq \Gamma_t$, $x^T_n \to x_t$ as $n \to \infty$.

Then

$$|x^T_n \vee y_t - x^T_m \vee y_t| \leq |x^T_n - x^T_m|$$

i.e. $\{x^T_n \vee y_t\}_{n=1}^{\infty}$ is a fundamental sequence and it is easy to prove that is convergent to $x_t \vee y_t$. 

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Thus, the operations will be defined for all pair $x, y$ such that $x, y \in X_\tau$, and there is no difficult to show that the order condition can be extended with $\Gamma_\tau$ on $X_\tau$.

3.8 Definition :[1]

An element from $X_+$ is called a Freudenthal unit and denoted by $\hat{i}$, if it follows from $x \in X, x \wedge \hat{i} = 0, \text{ that } x = 0$.

3.9 Proposition :

There exists a Freudenthal unit in $\hat{X}$ if and only if it exists in $X_\tau$ almost everywhere $\tau \in T$.

Proof :

Let $\hat{e} \in \hat{X}$ be a Freudenthal unit. Then from the equality $\hat{e} \wedge \hat{e} = 0$, we deduce that $\hat{e} = 0$. Moreover for all $\hat{x} \in \hat{G}$, we have $e(\tau) \wedge x = 0$, imply that $x = 0$ for all $x \in G_\tau$.

Since $G_\tau$ is dense in $K_\tau$, then, we get, that is true for all $x \in K_\tau$. It means that $e(\tau)$ is a Freudenthal unit in $X_\tau$ for almost everywhere $\tau \in T$.

Conversely, let $\{X_\tau\}_{\tau \in T} \subseteq S'(T, U)$, where $U$ is the Universal space, $\{X_\tau\}_{\tau \in T} \subseteq G_\tau$ for almost everywhere $\tau \in T$ and $x_\tau(\tau)$ be a Borel representation of elements in $\hat{G}$.

Then the set

$$A = \cap_{n=1}^\infty \{ (e, \tau) : \|x_\tau(\tau) \wedge e\|_\tau = 0 \text{ and } \|x_\tau(\tau)\|_\tau = 0 \}$$

is a Borelset, so since hypothesis

$$A_\tau = \{ e : e \text{ is a freudenthal unit in } X_\tau \} = \{ e : (e, \tau) \in A \} \neq \emptyset$$

for almost everywhere $\tau \in T$, then, there exist $e(\tau)$ such that $(e(\tau), \tau) \in A$ for almost everywhere $\tau \in T$. It is clear that $e = e(\tau)^n$ is a Freudenthal unit $\tilde{X}$.

References :-