Some Results on \((\sigma,\tau)\)-Left Jordan Ideals in Prime Rings

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Abstract

In this paper we have proved the following results. Let \(R\) be a prime ring, \(U\) be \((\sigma,\tau)\)-left Jordan ideal of \(R\) where \(\sigma,\tau: R\to R\) are two automorphisms of \(R\) and \(d\) be a nonzero derivation of \(R\). (1) If \((R,a)_{\sigma,\tau}\neq 0\), then \(a\in Z(R)\). (2) If \(aU=0\) or \(Ua=0\) and \(a\in R\), then \(U=0\) or \(U\subset Z(R)\). (3) If characteristic of \(R\) not equal 2 and \(U\subset C_{\sigma,\tau}\), then \(\sigma(u)+\tau(u)\in Z(R)\) for all \(u\in U\).

Proof:

By hypothesis \((R,a)_{\sigma,\tau}\neq 0\), then for all \(x,y\in R\), we have
\[0 = (xy,a)_{\sigma,\tau} = (xy)_{\sigma,\tau} - [x,\tau(a)]y = [x,\sigma(y)]y\]
by using hypothesis. So, we have \([x,\sigma(y)]y = 0\) for all \(x, y\in R\). Then \(x\sigma(y)\tau(a)\) for all \(x, y\in R\). Thus \(\tau^{-1}\) exists and \(\tau^{-1}(\sigma(y)\tau(a)\chi)y) = 0\). Also, \(\tau^{-1}\) is automorphism, then \(\tau^{-1}(x\sigma(y)\tau(a)\chi)y) = 0\). So, we get \([\tau^{-1}(x)\sigma(y)\tau(a)\chi)y = 0\) for all \(x, y\in R\). Thus \([\tau^{-1}(R),a] = 0\). So, we get \(a\in Z(R)\).

Theorem(2.2):

If \((R,a)_{\sigma,\tau}\neq 0\), then \(a\in Z(R)\).

Proof:

By hypothesis \((R,a)_{\sigma,\tau}\neq 0\), then for all \(x, y\in R\), we have
\[0 = (xy,a)_{\sigma,\tau} = (xy)_{\sigma,\tau} - [x,\tau(a)]y = [x,\sigma(y)]y\]
by using hypothesis. So, we have \([x,\sigma(y)]y = 0\) for all \(x, y\in R\). Then \(x\sigma(y)\tau(a)\) for all \(x, y\in R\). Thus \(\tau^{-1}\) exists and \(\tau^{-1}(\sigma(y)\tau(a)\chi)y) = 0\). Also, \(\tau^{-1}\) is automorphism, then \(\tau^{-1}(x\sigma(y)\tau(a)\chi)y) = 0\). So, we get \([\tau^{-1}(x)\sigma(y)\tau(a)\chi)y = 0\) for all \(x, y\in R\). Thus \([\tau^{-1}(R),a] = 0\). So, we get \(a\in Z(R)\).
0 = ax[y, σ(u)]. Then \( ax[y, σ(u)]=0 \) for all \( x, y \in R \), \( u \in U \). Since \( R \) is prime ring , we get \( a=0 \) or \( U \subset Z(R) \).

For another side if we have \( Ua=0 \), then for all \( x \in R \), \( u, v \in U \)

\[
0 = (xy, a)_{\alpha, \tau} = x \cdot (y, a)_{\alpha, \tau} - [x, τ(u)] ya
\]

Since \( τ \) is automorphism , we get

\[
[x, τ(u)] ya = 0 \quad \text{for all } x, y \in R, \quad u \in U \]

Theorem (2.3):

If \( R \) has a characteristic not equal to 2 and \( U \subset C_{\alpha, τ} \), then \( σ(u)+τ(u) \in Z(R) \) for all \( u \in U \).

Proof:

If \( σ, τ \) are any two automorphisms , then by the hypothesis, we have

\((v, u)_{\alpha, \tau} \in C_{\alpha, \tau} \) for all \( u, v \in U \). Then for all \( r \in R \), we get

\[
0 = [vσ(u)+τ(u) v, r]_{\alpha, τ}
\]

\[
= [vσ(u), r]_{\alpha, τ} + [τ(u) v, r]_{\alpha, τ}
\]

\[
= v[σ(u), r] + [τ(u)v, r]_{\alpha, τ}
\]

\[
= vσ([u, r]) + [τ(u)v, r]_{\alpha, τ}
\]

Since \( U \subset C_{\alpha, τ} \), then we have

\[
2vσ([u, r]) = 0 \quad \text{for all } u, v \in U, \quad r \in R.
\]

Also, we have \( R \) has a characteristic not 2 , then \( vσ([u, r]) = 0 \) for all \( u, v \in U, \quad r \in R \). Therefore, \( U \subset Z(R) \). Then we get \( σ(u)+τ(u) \in Z(R) \) for all \( u \in U \).

Theorem (2.4):

Let \( d(U)=0 \), \( dτ=dτ = dσ = σd \).

Then \( σ(u)+τ(u) \in Z(R) \) for all \( u \in U \).

Proof:

By the hypothesis \( d(U)=0 \), we have for all \( u \in U, x \in R \)

\[
0 = d((x, u)_{\alpha, τ}) = d(xσ(u)+ τ(u)x) = d(x)σ(u)+ xd(σ(u))+ d(τ(u))x + τ(u)d(x).
\]

Since \( dτ = dτ = dσ = σd \), we get

\[
dxσ(u) + τ(u)xd(x) = 0 \quad \text{for all } u \in U, \quad x \in R.
\]

That is \( (d(x), u)_{\alpha, τ} = 0 \) for all \( u \in U, \quad x \in R \).

So, we replace \( x \) by \( vx \), \( v \in U \). Then, we get

\[
0 = d((vx, u)_{\alpha, τ}) = d(vx)σ(u) + vxd(σ(u))+ d(τ(u))x + τ(u)d(x).
\]

Replace \( x \) by \( xv \), \( y \in R \). So, we have

\[
0 = [vτ(u) d(xy)= [vτ(u)] d(xy)+ [vτ(u)] yd(x)] \quad \text{for all } u, v \in U, \quad x, y \in R.
\]

Then \( [vτ(u)] yd(x) = 0 \) for all \( u, v \in U, \quad x, y \in R \). Thus \( [vτ(u)] yd(x) = 0 \) for all \( u, v \in U, \quad x, y \in R \). By a primeness of \( R \) and a non zero derivation \( d \) we get

\[
[vτ(u)] yd(x) = 0 \quad \text{for all } u, v \in U, \quad x, y \in R.
\]

For another side

\[
0 = (dy, u)_{\alpha, τ} = d(yv)σ(u) + vyd(σ(u))+ d(τ(u))x + τ(u)d(x).
\]

So by the same way we get \( [vσ(u)] = 0 \) for all \( u, v \in U \).

By the adding these relations (1) and (2), we get \( [vσ(u)] = 0 \) for all \( u, v \in U \). That is mean \( σ(u) + τ(u) \) in the center of \( R \). Then the center of \( U \) is subset of center of \( R \), then \( σ(u) + τ(u) \in Z(R) \) for all \( u \in U \).

References


الخلاصة

في هذا البحث تمكنا من برهنة النتائج التالية. ليكن حلقة أولبية 
$R$ ، وتكن $(\sigma, \tau)$ دوال متساوية في $R$.

1. إذا كان $(\sigma, \tau) = 0$، فان $(R \sigma) \cap (R \tau) = 0$.

2. إذا كان $a U = 0$ (أو $U a = 0$)، فان $a = 0$.

3. إذا كان $d(U) = 0$ واونه، $d(\tau) = \tau$ ، وانه $d(\sigma) = \sigma$.

4. إذا كان $u \in U$ لكي $Z(R) \ni \sigma(u) + \tau(u)$.