Certain Forms of $\beta^{**}$-Continuous Functions

Abstract:
In this work, we obtain new weak and strong forms of $\beta^{**}$-continuous functions Using the concept of $g\beta$-closed set.
We also obtain a characterization of $\beta^-T_1$ spaces.

المستخلص:
في هذا البحث قمنا أنماث ضعيفة وقوية من الدوال المستمرة $\beta^{**}$ باستخدام مفهوم المجموعات المغلقة $g\beta$ أيضا حصلنا تمييز للفضاءات $\beta^-T_1$.

1- Introduction:
In this paper, we introduce Weak form of $\beta^{**}$-continuous functions called $M$-$\beta^{**}$-continuous functions by using $g\beta$-closed sets obtain some basic properties of such functions also we introduce and study contra- $\beta^{**}$-continuous functions.
This notion is a stronger form of $M$-$\beta^{**}$-continuous functions.
Finally we introduce and study perfectly contra- $\beta^{**}$-continuous functions which is a strong form of $\beta^{**}$-continuous functions.
Throughout this paper $(X, \tau)$ and $(Y, \sigma)$ (or $X$ and $Y$) denote topological spaces on which no separation axioms are assumed unless explicitly stated.

2- Basic definitions:
In this section we recall the basic definitions needed in this work.

2-1 Definition:[1]
Let $(X, \tau)$ be a topological space, let $A \subseteq X$ then we say that:
i- $A$ is semi-open if $A \subseteq \text{cl} \text{ Int } A$. The complement of semi-open set is called semi-closed.
ii- $A$ is $\alpha$-open if $A \subseteq \text{Int} \text{ cl } \text{ Int } A$ The complement of $\alpha$-open set is called $\alpha$-closed.
iii- $A$ is $\beta$-open if $A \subseteq \text{cl} \text{ Int } \text{ cl } A$ The complement of $\beta$-open set is called $\beta$-closed.
2-2 Definition:[2]

i- the intersection of all semi-closed sets containing A is called the semi-closure of A and is denoted by $\text{Scl}A$

ii- the intersection of all $\alpha$-closed sets containing A is called the $\alpha$-closure of A and is denoted by $\alpha\text{ cl}A$.

iii- the intersection of all $\beta$-closed sets containing A is called the $\beta$-closure of A and denoted by $\beta\text{ cl}A$.

2-3 Definition:[1]

i- the family of all semi-open sets in X is denoted by $\text{SO}(X)$.

ii- the family of all $\alpha$-open sets in X is denoted by $\alpha\text{ O}(X)$.

iii- the family of all $\beta$-open sets in X is denoted by $\beta\text{ O}(X)$.

2-4 Definition:[2]

A subset F of $(X, \tau)$ is said to be:

i- g- closed in $(X, \tau)$ if $F \subseteq O$ and O is open $\Rightarrow \text{cl}(F) \subseteq O$.

ii- $g\alpha$- closed in $(X, \tau)$ if $F \subseteq O$ and O is $\alpha$- open $\Rightarrow \alpha\text{ cl}(F) \subseteq O$.

iv- gs- closed in $(X, \tau)$ if $F \subseteq O$ and O is open $\Rightarrow \text{scl}(F) \subseteq O$.

v- sg-closed in $(X, \tau)$, if $F \subseteq O$ and O is semi-open $\Rightarrow \text{scl}(F) \subseteq O$.

vi- $g\beta$-closed in $(X, \tau)$ if $F \subseteq O$ and O is $\beta$-open $\Rightarrow \beta\text{ cl}(F) \subseteq O$.

A subset W is said to be (g-open, $g\alpha$-open, gs-open, sg-open, $g\beta$-open ) if its complement $W^c=X-W$ is (g-closed, $g\alpha$-closed, gs-closed, sg-closed, $g\beta$-closed).

2-5 Definition:[3]

A function $f:(X, \tau)\rightarrow (Y, \sigma)$ is called:

i- $\beta^{**}$-continuous[3] if for each $v \in \beta\text{ O}(Y, \sigma)$ (that is v is $\beta$-open in Y) we have $f^{-1}(v)\in \beta\text{ O}(X, \tau).$(I.e. $f^{-1}(v)$ is $\beta$-open in X).

ii- $\beta^{**}$- closed if for every $\beta$- closed set W of $(X, \tau)$, $f(W)$ is $\beta$- closed in $(Y, \sigma)$.

iii- $\beta^{**}$-open if for every $\beta$-open set W of $(X, \tau)$, $f(W)$ is $\beta$-open in $(Y, \sigma)$.

iii- contra- $\beta$- closed if $f(U)$ is $\beta$-open in Y for each closed set U of X.

3- Main Results:

Before, we state the main results of this paper, we introduce the following definitions.

3-1 Definition:

A function $f:(X, \tau)\rightarrow (Y, \sigma)$ is said to be M-$\beta^{**}$-continuous if $\beta\text{ cl}(F) \subseteq f^{-1}(O)$ whenever O is a $\beta$-open subset of $(Y, \sigma)$, F is a $g\beta$-closed subset of $(X, \tau)$, and $F \subseteq f^{-1}(O)$. 168
3-2 Definition:
A function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is said to be \( M-\beta \)-closed if \( f(W) \subseteq \beta \text{int} A \) whenever \( A \) is a \( g\beta \)-open subset of \((Y, \sigma)\). \( W \) is a \( \beta \)-closed subset of \((X, \tau)\), and \( f(W) \subseteq A \).

3-3 Theorem:
i- \( f: (X, \tau) \rightarrow (Y, \sigma) \) is \( M-\beta^{**} \)-continuous if \( f^{-1}(O) \) is \( \beta \)-closed in \((X, \tau)\) for every \( \beta \)-open \( O \) in \((Y, \sigma)\), (that is if \( f \) is contra-\( \beta^{**} \)-continuous) [3].
ii- \( f: (X, \tau) \rightarrow (Y, \sigma) \) is \( M-\beta \)-closed if \( f(W) \) is \( \beta \)-open in \((Y, \sigma)\) for every \( \beta \)-closed subset \( W \) of \((X, \tau)\) (that is if \( f \) is contra-\( \beta^{**} \)-closed).

Proof:
i- Let \( F \subseteq f^{-1}(O) \), where \( O \) is \( \beta \)-open in \((Y, \sigma)\) and \( F \) is a \( g\beta \)-closed subset of \((X, \tau)\). Therefore \( \beta \text{cl}(F) \subseteq \beta \text{cl}(f^{-1}(O)) = f^{-1}(O) \). Thus \( f \) is \( M-\beta^{**} \)-continuous.

ii- Let \( f(W) \subseteq A \), where \( W \) is a \( \beta \)-closed subset of \((X, \tau)\) and \( A \) is a \( g\beta \)-open subset of \((Y, \sigma)\), therefore \( \beta \text{int}(f(W)) \subseteq \beta \text{int}(A) \) then \( f(W) \subseteq \beta \text{int}(A) \) thus \( f \) is \( M-\beta \)-closed.

3-4 Remark:
i- Clearly \( \beta^{**} \)-continuous functions are \( M-\beta^{**} \)-continuous.

ii- \( \beta^{**} \)-closed functions are \( M-\beta \)-closed.

Proof:
i- Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be \( \beta^{**} \)-continuous, let \( O \) be \( \beta \)-open in \((Y, \sigma)\), hence \( f^{-1}(O) \) is also \( \beta \)-open. Now \( F \) is \( g\beta \)-closed so \( F \subseteq f^{-1}(O) \Rightarrow \beta \text{cl}(F) \subseteq f^{-1}(O) \).

ii- Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be \( \beta^{**} \)-closed, let \( W \) be \( \beta \)-closed in \((X, \tau)\), so \( f(W) \) is also \( \beta \)-closed. Now \( A \) is \( g\beta \)-open so \( f(W) \subseteq A \Rightarrow f(W) \subseteq \beta \text{int}(A) \) (take the dual of the definition of \( g\beta \)-closed).

3-5 Theorem:
Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function from a space \((X, \tau)\) to space \((Y, \sigma)\):
i- let all subsets of \((X, \tau)\) be clopen, then \( f \) is \( M-\beta^{**} \)-continuous if and only if \( f \) is contra-\( \beta^{**} \)-continuous (that is \( f^{-1}(O) \) is \( \beta \)-closed in \((Y, \sigma)\)).

ii- Let all subsets of \((Y, \sigma)\) be clopen, then \( f \) is \( M-\beta \)-closed if and only if \( f \) is contra-\( \beta^{**} \)-closed (that is \( f(W) \) is \( \beta \)-open in \((Y, \sigma)\) for every \( \beta \)-closed subset \( W \) of \((X, \tau)\)).
Proof:
i- Assume that f is $M$-$\beta^{**}$-continuous. Let A be an arbitrary subset of $(X, \tau)$ such that $A \subseteq V$, where V is $\beta$-open in $(X, \tau)$ then by hypothesis $\beta \text{cl}(V) = V$ therefore all subsets of $(X, \tau)$ are g-$\beta$-closed (and hence all are g-$\beta$-open). So for any $O$ which is $\beta$-open in $(Y, \sigma)$, $f^{-1}(O)$ is g-$\beta$-closed in $(X, \tau)$. Since f is M-$\beta^{**}$-continuous, $\beta \text{cl}(f^{-1}(O)) \subseteq f^{-1}(O)$, therefore $\beta \text{cl}(f^{-1}(O)) = f^{-1}(O)$, i.e., $f^{-1}(O)$ is $\beta$-closed in $(X, \tau)$, so f is contra-$\beta^{**}$-continuous.

ii- Assume that f is M-$\beta$-closed as in (i), we obtain that all subsets of $(Y, \sigma)$ are g-$\beta$-open. Therefore for any $\beta$-closed subset W of $(X, \tau)$, $f(W)$ is g-$\beta$-open in Y. since f is M-$\beta$-closed, $f(W) \subseteq \beta \text{int}(f(W))$. Hence $f(W) = \beta \text{int}(f(W))$, i.e., $f(W)$ is $\beta$-open.

3-6 Corollaries:
Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function from a topological space $(X, \tau)$ to a topological space $(Y, \sigma)$:

i- Let all subsets of $(X, \tau)$ be clopen, then f is M-$\beta^{**}$-continuous if and only if f is $\beta^{**}$-continuous.

ii- Let all subsets of $(Y, \sigma)$ be clopen, then f is M-$\beta$-closed if and only if f is $\beta^{**}$-closed.

3-7 Example:
If $f : X \rightarrow Y$ is $\beta^{**}$-continuous, then f need not be contra-$\beta^{**}$-continuous, for example:
The identity function on the topological space $(X, \tau)$ where $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $X = \{a, b, c\}$, is $\beta^{**}$-continuous but not contra-$\beta^{**}$-continuous.

3-8 Definition:
A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called perfectly contra-$\beta^{**}$-continuous if the inverse of every $\beta$-open set in Y is $\beta$-clopen in X.

3-9 Remark:
Every perfectly contra-$\beta^{**}$-continuous function is contra-$\beta^{**}$-continuous and $\beta^{**}$-continuous.

3-10 Theorem:
If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\beta^{**}$-continuous function and M-$\beta$-closed, then $f^{-1}(A)$ is a g-$\beta$-closed whenever A is a g-$\beta$-closed subset of $(Y, \sigma)$.  

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Proof:
Let A be a $g\beta$-closed subset of $(Y, \sigma)$ suppose that $f^{-1}(A) \subseteq O$ where O is $\beta$-open in $(X, \tau)$, now $O^c \subseteq f^{-1}(A^c)$, so $f(O^c) \subseteq f^{-1}(A^c)$, but $f$ is $M\beta$-closed then $f(O^c) \subseteq \beta\text{int}(A^c) = (\beta\text{cl}(A))^c$. It follows that:

$O^c \subseteq f^{-1}(\beta\text{cl}(A))^c$ and hence: $f^{-1}(\beta\text{cl}(A)) \subseteq O$.

Since $f$ is $\beta^{**}$-continuous, $f^{-1}(\beta\text{cl}(A))$ is $\beta$-closed thus we have:

$\beta\text{cl}(f^{-1}(A)) \subseteq \beta\text{cl}(f^{-1}(\beta\text{cl}(A))) = f^{-1}(\beta\text{cl}(A)) \subseteq O$. This implies that $f^{-1}(A)$ is $g\beta$-closed in $(X, \tau)$.

3-11 Remark:
Let $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \gamma)$ be two functions such that $g \circ f: (X, \tau) \rightarrow (Z, \gamma)$ then:

i- $g \circ f$ is contra-$\beta^{**}$-continuous if $g$ is $\beta^{**}$-continuous and $f$ is contra-$\beta^{**}$-continuous.

ii- $g \circ f$ is contra-$\beta^{**}$-continuous if $g$ is contra-$\beta^{**}$-continuous and $f$ is $\beta^{**}$-continuous.

3-12 Theorem:
If $f: (X, \tau) \rightarrow (Y, \sigma)$ is $M\beta^{**}$-continuous and is an open, $g\beta$-closed subset of $(X, \tau)$, then the restriction $f_A = f|_{A}: (A, \tau_A) \rightarrow (Y, \sigma)$ is $M\beta^{**}$-continuous.

Proof:
Assume $F$ is a $g\beta$-closed subset relative to $A$ and $G$ is a $\beta$-open subset of $(Y, \sigma)$ for which $F \subseteq (f_A)^{-1}(G)$ then $F \subseteq f^{-1}(G) \cap A$.

On the other hand, $F$ is $g\beta$-closed in $X$, since $f$ is $M\beta^{**}$-continuous, then $\beta\text{cl}(F) \subseteq f^{-1}(G)$ this implies that $\beta\text{cl}(F) \cap A \subseteq f^{-1}(G) \cap A$, using that fact that $\beta\text{cl}(F) \cap A = \beta\text{cl}_{A}(F)$ [3].

We have: $\beta\text{cl}_{A}(F) \subseteq (f_A)^{-1}(G)$

Thus $f_A: (A, \tau_A) \rightarrow (Y, \sigma)$ is $M\beta^{**}$-continuous.

4- A Characterization of $\beta-T_{1/2}$ Spaces

In this section, we give a characterization of the class of $\beta-T_{1/2}$ spaces.
4-1 Definition:[4]
A space $(X, \tau)$ is said to be $\beta-T_\frac{1}{2}$ space, if every $g\beta$-closed set is $\beta$-closed.

4-2 Theorem:
Let $(X, \tau)$ be a space, then $(X, \tau)$ is a $\beta-T_\frac{1}{2}$ space if and only if $f : (X, \tau) \to (Y, \sigma)$ is $M-\beta^{**}$-continuous, for every space $(Y, \sigma)$ (and every function $f : (X, \tau) \to (Y, \sigma)$).

Proof:

$\Rightarrow$)
Let $F$ be a $g\beta$-closed subset of $(X, \tau)$ and $F \subseteq f^{-1}(O)$ where $O$ is $\beta$-open in $(Y, \sigma)$ since $(X, \tau)$ is a $\beta-T_\frac{1}{2}$ space $F$ is $\beta$-closed (i.e. $F=\beta\text{cl}(F)$)
Therefore $\beta\text{cl}(F) \subseteq f^{-1}(O)$ and hence $f$ is $M-\beta^{**}$-continuous.

$\Leftarrow$)
Let $W$ be a $g\beta$-closed subset of $(X, \tau)$ and $Y$ be the set $X$ with the topology $\sigma=\{\phi, Y, W\}$
Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function by assumption $f$ is $M-\beta^{**}$-continuous since $W$ is $g\beta$-closed in $(X, \tau)$ and $\beta$-open in $(Y, \sigma)$ and $W \subseteq f^{-1}(W)$, it follows that $\beta\text{cl}(W) \subseteq f^{-1}(W) = W$. Hence $W$ is $\beta$-closed in $(X, \tau)$, therefore $(X, \tau)$ is a $\beta-T_\frac{1}{2}$ space. $\Box$

References

(3) Mustafa Hadi J., Central- $\beta$-Continuous Functions First Kufa conference (2008), Kufa Univ.