The Endomorphism Ring Of Fully Stable Modules Relative To An Ideal

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ABSTRACT

Relatively concept has been used in mathematics specially in module theory. We study in early search a fully stable modules relative to an ideal which considered generalized of fully stable modules. The object of this work is to study the endomorphism rings of fully stable modules relative to an ideal.

INTRODUCTION

an $R$-module $M$ is said to be projective, if for every $R$-epimorphism $f : A \to B$ ($A$ and $B$ are $R$-modules) and every $R$-homomorphism $g : M \to B$, there exists an $R$-homomorphism $h : M \to A$ such that $g = f \circ h$ [1]. Zelmanowitz in [2] called an $R$-module regular if each finitely generated submodule is a projective direct summand. M. S. Abbas in [3] called a submodule $N$ of an $R$-module $M$ is stable if $\theta(N) \subseteq N$ for each $R$-homomorphism $\theta$ of $N$ into $M$. An $R$-module $M$ is fully stable if all its submodules are stable. A ring $R$ is stable if it is stable $R$-module. M.S. Abbas investigated the basic properties of this class of modules and he proved that if $M$ regular then, $M$ is fully stable iff $S = \text{End}_R(M)$ is commutative [3]. A submodule $N$ of an $R$-module $M$ is said to lie over a direct summand of $M$ if there exists a direct decomposition $M = P \oplus Q$ with $P \subseteq N$ and $N \cap Q$ small in $M$ [4]. Nicholson in [4] calls an $R$-module semi-regular if every cyclic submodule lies over a projective direct summand. In [5] a generalization of fully stable modules, namely fully stable modules relative to an ideal is introduced. An $R$-module $M$ is called fully stable relative to an ideal $A$ of $R$, if $\theta(N) \subseteq N + MA$ for each submodule $N$ of $M$ and each $R$-homomorphism $\theta : N \to M$ [5]. It is an easy matter to see that $M$ is fully stable, if and only if $\theta(xR) \subseteq xR$ for each $x$ in $M$ and $R$-homomorphism $\theta : xR \to M$ [3]. And in [5] we proved, an $R$-module $M$ is fully stable relative to an ideal $A$ of $R$ if and only if for each $x, y$ in $M$ with $y \notin xR + MA$ implies $\text{ann}_R(x) \not\subseteq \text{ann}_R(y)$ if and only if $\text{ann}_M(\text{ann}_R(x)) \subseteq \langle x \rangle + MA$, for each $x$ in $M$. In this work, we study the
endomorphism ring of fully stable modules relative to an ideal. A necessary condition for fully stable relative to an ideal of the endomorphism ring for modules is studied, that is, if $M$ be an $R$-module, $S = \text{End}_R(M)$ and $A$ be an ideal of $S$ and $S$ is a right fully stable ring relative to $A$, then $\ker(\beta) \subseteq \ker(\gamma)$ implies that $\gamma \in S\beta + SA$, for all $\beta, \gamma \in S$.

Throughout this paper $R$ an arbitrary commutative ring with unity and all modules will be unitary right $R$-module. We use the notation $\leq$ for the submodules. The right (resp. left) annihilater of a subset $X$ of arbitrary ring $R$ (not necessary commutative) is denoted by $r-\text{ann}_R(X)$ (resp. $\ell-\text{ann}_R(X)$). For an $R$-module $M$, $S = \text{End}_R(M)$ and $E(M)$ will respectively stand for the endomorphism ring of $M$ and injective envelop of $M$.

**RESULTS**

Let $M$ be an $R$-module we say that $S$ is commutative modulo $MA$ (where $A$ is an ideal of $R$) if $(f \circ g)(x) = (g \circ f)(x) \in MA$ for each $f, g$ in $S$ and $x \in M$. It is clear that if $S$ is a commutative ring, then $S$ is a commutative modulo $MA$ for each non-zero ideal $A$ of $R$. In fact, $S$ is commutative if and only if $S$ is commutative modulo $MA$ (where $A$ is the zero ideal). It is proved in [3] that, if $M$ is a fully stable $R$-module, then $\text{End}_R(M)$ is a commutative ring. For fully stable modules relative to an ideal we have the following.

**Proposition (2.1):** If $M$ is fully stable $R$-module relative to an ideal $A$ of $R$, then $S = \text{End}_R(M)$ is commutative modulo $MA$.

**Proof:** Let $f, g \in \text{End}_R(M)$ and $x \in M$. Since $M$ is a fully stable module relative to $A$, $f(xR) \subseteq xR + MA$ and $g(xR) \subseteq xR + MA$, by [4, Theorem (3.2)], thus there exist $r_1, r_2 \in R$ and $w_1, w_2 \in MA$ s.t. $f(x) = xr_1 + w_1$ and $g(x) = xr_2 + w_2$. Hence, $(f \circ g)(x) - (g \circ f)(x) = r_1r_2 + w_1r_2 + f(w_2) - x(r_2r_1 - w_2r_1 - g(w_1)).$ $\square$

The converse of Proposition (2.1), need not be true in general. For example, consider the ring $Z$ of integers. Since $\text{End}_Z(R) \cong R$ [1], thus the endomorphism ring of $Z$ is commutative modulo $nZ$ for each $n$ in $Z$, but $Z$ is not fully stable relative to each proper ideal of $Z$ [5, (2.3)(3)].

Recall that, a submodule $N$ of an $R$-module $M$ is said to lie over a direct summand of $M$ if there exists a direct decomposition $M = P \oplus Q$ with $P \subseteq N$ and $N \cap Q$ small in $M$ [4]. an $R$-module $M$ is said to be
projective, if for every $R$-epimorphism $f: A \to B$ ($A$ and $B$ are $R$-modules) and every $R$-homomorphism $g: M \to B$, there exists an $R$-homomorphism $h: M \to A$ such that $g = f \circ h$ [1]. Nicholson in [4] calls an $R$-module semi-regular if every cyclic submodule lies over a projective direct summand.

For the converse of Proposition (2.1), we suggest the following definition.

**Definition (2.2):** A submodule $N$ of an $R$-module $M$ is said to lie over a direct summand relative to an ideal $A$ of $R$ if there exists a direct decomposition $M = P \oplus Q$ such that $P \subseteq N$ and $N \cap Q \subseteq MA$. $M$ is said to be semi-regular relative to $A$ if every cyclic submodule lies over a direct summand relative to $A$.

Every direct summand of an $R$-module lies over direct summand relative to zero ideal and hence lies over direct summand relative to each non-zero ideal of $R$. The converse needs not be true. For example, consider $Z_{12}$ as $Z_{12}$-module, it is clear that $H = \{0, 2, 4, 6, 8, 10\}$ is not direct summand, while $H$ lies over direct summand relative to itself. Also, if every cyclic submodule of finitely generated $R$-module $M$ lies over projective direct summand relative to small ideal $A$ of $R$, then $M$ is a semi-regular.

**Examples and Remarks (2.3):**
1. Recall that, an $R$-module $M$ is said to be regular if each finitely generated submodule is a projective direct summand [2]. If $M$ is semi-regular $R$-module relative to an ideal, then $M$ need not be regular module. For example, $Q$ as $Z$-module is semi-regular relative to each non-zero ideal, while it is not regular module.
2. Recall that, An $R$-module $M$ is called principally quasi-injective (shortly, PQ-injective) if each $R$-homomorphism from a principal submodule of $M$ into $M$ can be extended to an $R$-endomorphism of $M$ [6]. It is clear that, if every cyclic submodule of an $R$-module $M$ is a direct summand, then $M$ is PQ-injective [6], and semi-regular relative to each ideal.
3. Every regular $R$-module $M$ is PQ-injective and semi-regular relative to each ideal. Since in regular module every cyclic submodule is a direct summand [2], thus $M$ is PQ-injective and semi-regular relative to each ideal by using (2).
4. $Q$ as $Z$-module is semi-regular relative to each non-zero ideal of $Z$. But it is not semi-regular relative to zero ideal.
5. There is no direct relation between semi-regular \( R \)-module relative to an ideal \( A \) of \( R \) and fully stable \( R \)-module relative \( A \). For example, since every vector space \( V_F \) of dimension \( n \), for each \( n > 1 \) is a semi-simple module, then by using (2), \( V_F \) is a semi-regular relative to zero ideal of \( F \), while \( V_F \) is not fully stable relative to zero ideal of \( F \) [5, (2.3)(14)]. On the other hand, \( Z_{12} \) as \( Z_{12} \)-module is a fully stable relative to each ideal in particular, it is fully stable relative to \( H = \{0,4,8\} \), [5, Corollary (3.5)]. But, it is not semi-regular relative to \( H \). Since there exists principal ideal \( K = \{0,2,3,6,8,10\} \), which does not lie over direct summand relative to \( H \).

For the converse of Proposition (2.1), we have the following.

**Proposition (2.4):** Let \( M \) be \( PQ \)-injective \( R \)-module and \( M \) be semi-regular relative to an ideal \( A \) of \( R \). If \( \text{End}_R(M) \) is commutative modulo \( MA \), then \( M \) is a fully stable module relative to \( A \).

**Proof:** Let \( N \) be a cyclic submodule of \( M \) and \( f : N \to M \) be any \( R \)-homomorphism. By hypothesis, there exists a direct decomposition \( M = P \oplus Q \) with \( P \subseteq N \) and \( Q \cap N \subseteq MA \). Since \( M \) is \( PQ \)-injective, then there exists an \( R \)-homomorphism \( g : M \to M \) such that \( g|_N = f \). Now, let \( \rho \) be the projection map of \( M \) onto \( P \). For each \( x \) in \( N \), \( x = p + q \), for some \( p \in P \) and \( q \in Q \cap N \). Then \( f(x) = y + l \), for some \( y \in P \) and \( l \in Q \). Hence, \((\rho \circ g)(x) = \rho(g(x)) = \rho(f(x)) = \rho(y + l) = y \). On the other hand, \((g \circ \rho)(x) = g(\rho(x)) = g(\rho(p + q)) = g(p) = f(p) = y + l - f(q)\). Since the endomorphism ring of \( M \) is commutative modulo \( MA \), then \( l - f(q) = y + l - f(q) - y = (g \circ \rho)(x) - (\rho \circ g)(x) \in MA \). Since \( l - g(q) = l - f(q) \), \( q \in MA \) and \( MA \) is fully invariant submodule of \( M \), we have \( l \in MA \). Thus \( f(x) - y \in MA \), so \( f(x) \in P + MA \subseteq N + MA \). Therefore, \( N \) is stable submodule relative to \( A \).

It is proved in [3] that, if \( M \) be an \( R \)-module in which every cyclic submodule is a direct summand and \( S = \text{End}_R(M) \) is commutative, then \( M \) is a fully stable module.

For fully stable modules relative to an ideal we have the following, which is follows from Remark (2.3)(2).

**Corollary (2.5):** Let \( M \) be an \( R \)-module in which every cyclic submodule is a direct summand and \( A \) be an ideal of \( R \). If \( S = \text{End}_R(M) \) is commutative modulo \( MA \), then \( M \) is a fully stable relative to \( A \).
From Proposition (2.1) and Proposition (2.4), we have the following.

**Corollary (2.6):** Let $M$ be a PQ-injective $R$-module and semi-regular relative to an ideal $A$ of $R$. Then $M$ is fully stable module relative to $A$ if and only if $S = \text{End}_R(M)$ is commutative modulo $MA$. □

From above corollary with Remark (2.3)(3), we have the following corollary which is a generalization of that on fully stable module, if $M$ be a regular $R$-module. Then $M$ is fully stable module if and only if $\text{End}_R(M)$ is commutative [3].

**Corollary (2.7):** Let $M$ be a regular $R$-module. Then the following statements are equivalent for an ideal $A$ of $R$.
1. $M$ is fully stable module relative to $A$.
2. $S$ is commutative modulo $MA$. □

**Proposition (2.8):** Let $M$ be PQ-injective $R$-module in which every cyclic submodule lies over a direct summand relative to an ideal $A$ of $R$. If $S$ is a right fully stable ring relative to $K = \text{Hom}_R(M, MA)$, then $M$ is a fully stable module relative to $A$.

**Proof:** Let $xR$ be a cyclic submodule of $M$ and $\alpha : xR \to M$ be any $R$-homomorphism. Let $I = \text{Hom}_R(M, xR)$. It is clear that $I$ is a right ideal of $S$. Define $\theta : I \to S$ by $\theta(\beta) = \alpha \circ \beta$, for each $\beta \in I$. It is clear that $\theta$ is a well-defined and it is easily matter to see that $\theta$ is an $S$-homomorphism. Now, since $S$ is a fully stable ring relative to $K$, then $\theta(I) \subseteq I + KS = I + K$ and by hypothesis, there exists a direct decomposition $M = P \oplus Q \ni P \subseteq xR$ and $Q \cap xR \subseteq MA$. Then $x = p + q$, for some $p$ in $P$ and $q$ in $Q \cap xR$. Now, since $M$ is PQ-injective, then $\alpha$ can be extended to $R$-endomorphism of $M$ say $g$. The natural projection $\rho$ of $M$ onto $P$ belong to $I$. Hence, $\alpha(x) = \alpha(p + q) = \alpha(p) + \alpha(q) = \alpha(\rho(p)) + g(q) = (\theta(\rho))(p) + g(q)$. And since $\theta(\rho) \in I + K$, thus there exists $f \in I$ and $h \in K \ni (\theta(\rho))(p) = f(p) + h(p) \in xR + MA$ and since $MA$ is a fully invariant submodule of $M$, then $g(q) \in MA$. Hence, $\alpha(x) \in xR + MA$. □

From Remark (2.3)(2) with Proposition (2.8), we have the following which is a generalization of that, if $M$ be an $R$-module in which every cyclic submodule is a direct summand and $\text{End}_R(M)$ is commutative, then $M$ is a fully stable module [3, Lemma (2.5), p. 18].
**Corollary (2.9):** Let $M$ be an $R$-module in which every cyclic submodule is a direct summand and $A$ be an ideal of $R$. If $S = \text{End}_S(M)$ is a fully stable ring relative to $K = \text{Hom}_S(M, MA)$, then $M$ is a fully stable module relative to $A$. □

**Corollary (2.10):** Let $M$ be a PQ-injective $R$-module and semi-regular relative to an ideal $A$ of $R$. If $S = \text{End}_S(M)$ is a fully stable ring relative to $K = \text{Hom}_S(M, MA)$, then $M$ is a fully stable module relative to $A$. □

From Remark (2.3)(3) with above corollary, we have the following which is a generalization of that on fully stable module, every regular module with fully stable ring of endomorphism is a fully stable module [3, Corollary (2.6), p. 18].

**Corollary (2.11):** Let $M$ be a regular $R$-module and $A$ be an ideal of $R$. If $S = \text{End}_S(M)$ is a fully stable ring relative to $K = \text{Hom}_S(M, MA)$, then $M$ is a fully stable module relative to $A$. □

Recall that, if $M$ and $B$ are two $R$-modules. $B$ generates $M$ if $M = \sum \text{Im}(\phi)$. $B$ cogenerates $M$ if $0 = \bigcap \ker(\phi)$ [1].

In the following we study necessary condition for full stability relative to an ideal of endomorphism rings.

**Proposition (2.12):** Let $M$ be an $R$-module generates $\ker(\beta)$ for each $\beta \in S = \text{End}_S(M)$ and $A$ be an ideal of $S$. Then, $S$ is a right fully stable ring relative to $A$, iff if $\ker(\beta) \subseteq \ker(\gamma)$ implies that $\gamma \in S\beta + SA$, for all $\beta, \gamma \in S$.

**Proof:** ($\Rightarrow$). For each $\alpha \in r - \text{ann}_S(\beta)$, $\beta \alpha = 0$, so $\text{Im}(\alpha) \subseteq \ker(\beta)$, hence $\text{Im}(\alpha) \subseteq \ker(\gamma)$, thus $\gamma \alpha = 0$, that is $\alpha \in r - \text{ann}_S(\gamma)$. By [5, Proposition (2.8)] we have $\gamma \in S\beta + SA$.

($\Leftarrow$). If $\gamma \in \ell - \text{ann}_S(r - \text{ann}_S(\beta))$, we show that $\ker(\beta) \subseteq \ker(\gamma)$. For any $x \in \ker(\beta)$, $x = \sum_{i=1}^t \alpha_i(m_i)$ where $\alpha_i : M \to \ker(\beta)$ and $m_i \in M$, $i = 1, \ldots, t$. Thus $\beta \alpha_i = 0$, $i = 1, \ldots, t$, so $\gamma \alpha_i = 0$, $i = 1, \ldots, t$. It is follows that $x \in \ker(\gamma)$. By hypothesis $\gamma \in S\beta + SA$. [5, Corollary (3.5)] completes the proof. □

As dual results of Proposition (2.12) we have the following.
**Proposition (2.13):** Let $M$ be an $R$-module which is a cogenerate of $M/\beta(M)$ for each $\beta \in S = \text{End}_{\alpha}(M)$ and $A$ be an ideal of $S$. Then $S$ is a right fully stable ring relative to $A$, iff if $\gamma(M) \subseteq \beta(M)$ implies that $\gamma \in S\beta + SA$, for each $\gamma, \beta \in S$.

**Proof:** $(\Rightarrow)$. Let $\alpha \in \ell - \text{ann}_S(\beta)$, then $\alpha\beta = 0$ so $\text{Im}(\beta) \subseteq \text{ker}(\alpha)$, then $\text{Im}(\gamma) \subseteq \text{ker}(\alpha)$ and hence $\alpha \in \ell - \text{ann}_S(\gamma)$. By [5, Proposition (2.8)], $\gamma \in S\beta + SA$.

$(\Leftarrow)$. Let $\gamma \in r - \text{ann}_S(\ell - \text{ann}_S(\beta))$, we show that $\gamma(M) \subseteq \beta(M)$. If not, then $\exists m_0 \in M \ni \gamma(m_0) \notin \beta(M)$, thus the natural epimorphism $\pi : M \to M/\beta(M)$ is a non-zero. Then $\exists \sigma : M/\beta(M) \to M$ with $\sigma(\gamma(m_0) + \beta(M)) \neq 0$. Define $f : M \to M$ by $f(m) = \sigma(m + \beta(M))$. Therefore $\sigma$ is a well-defined $R$-homomorphism. Then $(f \circ \gamma)(m_0) = \sigma(\gamma(m_0) + \beta(M)) \neq 0$, while $f\beta(m) = \sigma(\beta(m) + \beta(M)) = \sigma(\beta(M)) = 0$, $\forall m \in M$. Thus $f\gamma \neq 0$ and $f\beta = 0$, which is a contradiction. Hence, by hypothesis $\gamma \in S\beta + SA$. Therefore $S$ is a right fully stable ring relative to $A$, [5, Corollary (3.5)]. □

Notes that, the condition in Proposition (2.12) and (2.13), not used in the necessary condition.

Recall that an $R$-module $M$ is distinguished if $\text{ann}_R(I) \neq 0$, for all maximal ideal $I$ of $R$ [7]. Then the following statements are equivalent for an $R$-module $M$.

1. $M$ is distinguished.
2. $M$ contains a copy of every simple $R$-module.
3. Every non-zero finitely generated $R$-module is dualizable with respect to $M$ (that is $U^\ast = \text{Hom}_R(U,M) \neq 0$ for every finitely generated $R$-module $U$) [7]. □

Recall that the module $C$ is a cogenerator of $\text{mod}-R$ if for each $M$ in $\text{mod}-R$, $0 = \bigcap_{\phi \in \text{Hom}(M,C)} \text{ker}(\phi)$ [1].

The following theorem appears in [7].

**Theorem (2.14):** Let $M$ be an $R$-module. Then $M$ is distinguished if and only if $E(M)$ is a cogenerator for $\text{mod}-R$. □

It follows from the above theorem that if $M$ is injective $R$-module, then $M$ is distinguished if and only if it is a cogenerator for $\text{mod}-R$ [7].

**Theorem (2.15):** Let $M$ be a distinguished $R$-module and $A$ be an ideal of $S' = \text{End}_{\alpha}(E(M))$. Then $S'$ is a right fully stable ring relative to $A$ if and only if $\gamma(E(M)) \subseteq \beta(E(M))$ implies that $\gamma \in S'\beta + SA$ for all $\gamma, \beta \in S'$. 197
Proof: \((\Rightarrow)\). Suppose that \(S'\) is a fully stable ring relative to \(A\) and \(\gamma, \beta \in S'\) with \(\gamma(E(M)) \subseteq \beta(E(M))\) by the same proof of Proposition (2.13), with replacing \(M\) by \(E(M)\) and \(S\) by \(S'\) we have that, \(\gamma \in S'\beta + S'\).

\((\Leftarrow)\). Since \(M\) is distinguished \(R\)-module, then by Theorem (2.14), \(E(M)\) cogenerates for \(\text{mod-}R\), in particular \(E(M)\) cogenerates \(E(M)/\beta(E(M)), \forall \beta \in S'\). By hypothesis and the same proof of Proposition (2.13), with replacing \(M\) by \(E(M)\) and \(S\) by \(S'\) we have that, \(S'\) is a fully stable ring relative to \(A\). □

**Corollary (2.16):** Let \(M\) be injective distinguished \(R\)-module and \(A\) be an ideal of \(S\). Then \(S\) is a right fully stable ring relative to \(A\) if and only if \(\gamma(M) \subseteq \beta(M)\) implies that \(\gamma \in S\beta + SA\) for all \(\gamma, \beta \in S\). □

**Proof:** By using Theorem (2.15) and \(E(M) = M\) (because \(M\) is an injective \(R\)-module). □

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