\section*{\textit{G}-cyclicity}

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\section*{Abstract}

Dimensional separable complex Hilbert space and \(S\) be a multiplication semigroup of \(L^1\) with \(1\). Generalizing the concept of supercyclicity, we define \(G\)-cyclicity, namely, an operator \(T\) is called \(G\)-cyclic over \(S\) if there is a vector \(x \in H\) such that \(\{\alpha T^n x \mid \alpha \in S, n \geq 0\}\) is norm-dense in \(H\), such a vector is called \(G\)-cyclic vector for \(T\) over \(S\). In this paper, we list some basic properties of \(G\)-cyclicity. We study necessary and sufficient conditions for an operator to be \(G\)-cyclic. Finally we study some of the spectral properties of \(G\)-cyclic operators.

\section*{Introduction}

Let \(H\) be an infinite - dimensional separable complex Hilbert space, and let \(B(H)\) be the Banach algebra of all linear bounded operators on \(H\). Let \(T\) be an element in \(B(H)\), \(T\) is called "cyclic" if there is a vector \(x \in H\) such that the closed linear space spanned by the set \(\{T^k x \mid k \geq 0\}\) is the whole space \([5,p88]\). An operator \(T\) is "hypercyclic" if there is a vector \(x \in H\) such that the set \(\{T^k x \mid k \geq 0\}\) is norm-dense in \(H\) \([1, p71]\), and \(T\) is called "supercyclic" if there exists an \(x \in H\) such that \(\{\alpha T^k x \mid k \geq 0, \alpha \in \mathbb{C}\}\) is norm-dense in \(H\) \([8]\).

In this paper we introduce a concept which unifies these concepts as follows: Let \(T\) an element in \(B(H)\), and let \(A = A(T)\) be the subalgebra of \(B(H)\) generated by the identity operator \(I\) and the operator \(T\) over the field of complex numbers. It is easily seen that \(A\) is the set of all polynomials in \(T\) with coefficients in \(\mathbb{C}\). Let \(G = G(T)\) be a multiplication semigroup of \(A\) with \(1\). In particular, let \(S\) be a multiplication semigroup of \(\mathbb{C}\) with \(1\), and let \(G = G(S,T)\) be the multiplication semigroup of \(A\) consisting of all elements in \(A\) of the form \(\{\alpha T^n \mid \alpha \in S, n \geq 0\}\) or \(G = A\).

Let \(x\) be a vector in \(H\), we say that \(x\) is a \(G\)-cyclic vector over \(S\) for \(T\) if \(\{gx \mid g \in G\}\) is norm-dense in \(H\). We call such phenomena cyclic phenomena, we point out that this term is used in the literature to mean the following three cases:

1. \(G = A\) (Cyclicity).
2. \(S = \emptyset\) and \(G = G(\emptyset,T)\) (Supercyclicity).
3. \(S = \{1\}\) and \(G = G(\{1\},T)\) (Hypercyclicity).

In this paper we restrict our study on a multiplication semigroup of \(\mathbb{C}\) with \(1\). Clearly, every hypercyclic operator is \(G\)-cyclic, and every \(G\)-cyclic operator is supercyclic.

In \([9],[10]\) the authors studied \(G\)-cyclicity when, \(S = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}\), \(S = B^* = \{\lambda \in \mathbb{C} \mid |\lambda| \geq 1\}\). An example was given for an operator which \(G\)-cyclic over \(B^*\) but not over \(B^*\) and vice versa.

This paper consists of three sections. In section one; we list basic properties of \(G\)-cyclicity. In section two, we give necessary and sufficient conditions for \(G\)-cyclicity. In section three, we investigate the spectrum of \(G\)-cyclic operators.

\section*{Preliminaries}

In this section we introduce the definition of \(G\)-cyclicity and give some basic properties.

\section*{Definition}

Let \(S\) be a multiplication semigroup of \(\mathbb{C}\) with \(1\), an operator \(T\) on a separable complex Hilbert space \(H\) is called \(G\)-cyclic over \(S\) if there is a vector \(x\) in \(H\) such that the set \(\{\alpha T^n x \mid \alpha \in S, n \geq 0\}\) is norm-dense in \(H\). In this case \(x\) is called a \(G\)-cyclic vector for \(T\) over \(S\). It is clear that we may assume \(0 \in S\).

Next we fix notation required for the discussion.
Notation. Let $S$ be a multiplication semigroup of $H$ with $1$, and $T \in B\left(\mathcal{H}\right)$. 
1. $\mathcal{GC}_{S}(T) = \{ x \in H | x \text{ is a } \mathcal{G}-\text{cyclic vector for } T \text{ over } S \}$
2. $\mathcal{GC}_{S}\left(\mathcal{I}_{N}(T)\right) = \{ x \in B(\mathbb{C}^{N}) | x \text{ is } \mathcal{G}-\text{cyclic operator over } S \}$
3. $\text{Sorbt} \left( T, x \right) = \{ x \alpha^{n} | x \in S, \alpha \in S, \alpha \geq 0 \}$
4. $S = \{ x | x \in S \}$
5. $S^{-1} = \{ x^{-1} | x \in S \}.

Note that \( S^{-1} \) is a semigroup of $H$ with $1$ and \( \left(S^{-1}\right)^{-1} = S \).

Remarks.
1. $x \in \mathcal{GC}_{S}(T)$ if and only if $\text{Sorbt} \left( T, x \right) = \{ 0 \}$.
2. Clearly, from (1.1) every hypercyclic operator (vector) is $\mathcal{G}$-cyclic, and every $\mathcal{G}$-cyclic is supercyclic.

Since the range of a superacyclic operator $R(T)$ is dense [3], then we have:\
6. Proposition. The range of a $\mathcal{G}$-cyclic operator $T$ on $H$ is a dense in $H$.

We begin with an easy observation. Compare the following result with [7].
7. Proposition. Let $x \in \mathcal{GC}_{S}(T)$, then

$$\inf \{ y \left\| T^{n}x \right\| | n \geq 0, y \in S \} = 0$$
and

$$\sup \{ y \left\| T^{n}x \right\| | n \geq 0, y \in S \} = \infty.$$ 

Proof. Let $x \in \mathcal{GC}_{S}(T)$, and assume that

$$\inf \{ y \left\| T^{n}x \right\| | n \geq 0, y \in S \} = m > 0.$$ 

Since $0 \in H$, then there are sequences $\{\alpha_{k}\}$ in $S$ and $\{n_{k}\}$ in $N$ such that $\left\| T^{n_{k}}x \right\| \to 0$. Hence there is $n_{k}$ such that $\left\| T^{n_{k}}x \right\| < m$ for all $j > k$, a contradiction.

Now assume that

$$\sup \{ y \left\| T^{n}x \right\| | n \geq 0, y \in S \} = M < \infty.$$ 

Then $y \in H$ such that $\left\| y \right\| > M$. Since $x \in \mathcal{GC}_{S}(T)$, then there exist sequences $\{n_{k}\}$ in $N$, $\{\alpha_{k}\}$ in $S$ such that $\left\| \alpha_{k} \left\| T^{n_{k}}x \right\| \to \left\| y \right\|$. Thus we get a sequence of

$$\left\{ y \left\| T^{n}x \right\| | n > 0, y \in S \right\}$$
that converges to $y$.

Then $\left\| y \right\| \leq M_{\left\| T \right\|}$, a contradiction. \(\Box\)

For special case, when $S$ is bounded, one can easily prove the following corollaries.

8. Corollary. If $S$ is a bounded semigroup and $x \in \mathcal{GC}_{S}(T)$, then

$$\sup \{ \left\| T^{n}x \right\| | n \geq 0 \} = \infty.$$ 

9. Corollary. Let $T \in B\left(\mathcal{H}\right)$, and $S$ be a bounded semigroup if $\left\| T \right\| < 1$, then $T \in \mathcal{GC}_{S}\left(\mathcal{H}\right)$.

Remark. The backward shift $B : \ell^{2}(\mathbb{C}) \to \ell^{2}(\mathbb{C})$ is not $\mathcal{G}$-cyclic over any bounded semigroup, since $\left\| B \right\| = 1$.

One can prove easily the following results that implies if $x \in \mathcal{GC}_{S}(T)$, then $\alpha T^{n}x \in \mathcal{GC}_{S}(T)$ for all $\alpha \in S$ and $n \geq 0$.

10. Proposition. Let $x \in \mathcal{GC}_{S}(T)$, and let $\mathcal{F} \in B\left(\mathcal{H}\right)$ such that $\mathcal{F}T = T\mathcal{F}$ and $R\left(\mathcal{F}\right)$ is dense in $\mathcal{H}$.

Then $\mathcal{F}x \in \mathcal{GC}_{S}(T)$.

Terming now to the similarity of $\mathcal{G}$-cyclic operators.

11. Proposition. Let $H, K$ be Hilbert spaces, let $T \in B\left(\mathcal{H}\right)$ and $\mathcal{F} \in B\left(K\right)$, and let $\mathcal{X} : H \to K$ be a bounded linear transformation such that $R\left(\mathcal{X}\right)$ is dense in $K$, and $\mathcal{F}\mathcal{X} = \mathcal{X}T$. If $T \in \mathcal{GC}_{S}(H)$, then $\mathcal{F} \in \mathcal{GC}_{S}(K)$. In particular, if $T, \mathcal{F} \in B\left(\mathcal{H}\right)$ are similar operators, then $T \in \mathcal{GC}_{S}(H)$ if and only if $\mathcal{F} \in \mathcal{GC}_{S}(\mathcal{H})$.

Proof. Let $y \in \mathcal{GC}_{S}(T)$, then $\text{Sorbt} (T, y)$ is dense in $H$, thus

$$\min \left\{ y \left\| T^{n}x \right\| | n \geq 0, y \in S \right\}.$$

Next we turn our attention to the direct sum of $\mathcal{G}$-cyclic operators. Compare with [9]. The proof is left to the reader.
12. Proposition. Let \( \{ H_i \} \) be family of Hilbert spaces, let \( T_i \in B(H_i) \) for all \( i \). If 
\[ \bigoplus_i U_i \in \mathcal{G}_S \left( \bigoplus_i H_i \right), \]
then 
\[ T_i \in \mathcal{G}_S \left( H_i \right) \quad \text{for all} \quad i. \]

1. Necessary and Sufficient Conditions for \( G \)-Cyclicity

The goal of this section is to give a characterization for \( G \)-cyclic operators. We first characterize the set of all \( G \)-cyclic vectors.

2.1. Proposition. Let \( T \in B(H) \).

Then 
\[ \mathcal{G}_S(T) = \bigcap \left( \bigcup_{\alpha \in S, n \in \mathbb{N}} \left( \frac{1}{\alpha} U_k \right) \right), \]
where \( \{ U_k \}_{k=1}^\infty \) is a countable base for the topology on \( H \).

Proof. Since \( H \) is separable, then let \( \{ U_k \}_{k=1}^\infty \) be a countable base for the topology on \( H \), \( x \in \mathcal{G}_S(T) \) if and only if 
\[ \{ \alpha^n T^n x | x \in S, n \geq 0 \} \text{ is dense in } H, \text{ if and only if } \forall k \geq 1, \exists \alpha \in S, n \in \mathbb{N}, \text{ such that } \alpha^n T^n x \in U_k, \text{ if and only if } \forall k \geq 1, \exists \alpha \in S, n \in \mathbb{N}, \text{ such that } T^n x \in \frac{1}{\alpha} U_k, \]
then 
\[ \forall x \in \bigcap \left( \bigcup_{\alpha \in S, n \in \mathbb{N}} \left( \frac{1}{\alpha} U_k \right) \right). \]
Recall that a countable intersection of open sets is \( G \)-set.

2.2. Corollary. If the set of \( G \)-cyclic vectors over \( S \) is not empty, then it is \( G \)-set in \( H \).

Proof: Let \( x \in \mathcal{G}_S(T) \). By \( (1.7) \) 
\[ \alpha^n T^n x \in \mathcal{G}_S(T) \text{ for all } n \geq 0, \alpha \in S. \]
Thus \( S \circ (T, x) \subseteq \mathcal{G}_S(T) \), hence \( \mathcal{G}_S(T) \) is dense in \( H \). Now by \( (2.1) \)
\[ \mathcal{G}_S(T) = \bigcap \left( \bigcup_{\alpha \in S, n \in \mathbb{N}} \left( \frac{1}{\alpha} U_k \right) \right), \]
where \( \{ U_k \}_{k=1}^\infty \) is a countable base for the topology on \( H \). The desired result follows by the continuity of \( T \) and because \( \frac{1}{\alpha} U_k \) is open for all \( k \), and all \( \alpha \in S \). \( \Box \)

The following result is a characterization for \( G \)-cyclic operators, compare with [4].

2.3. Theorem. Let \( T \in R(H) \). The following statements are equivalent:

1. \( T \in \mathcal{G}_S(H) \).

2. For each non-empty open sets \( U, V \), there are \( \alpha \in S, n \in \mathbb{N} \) such that 
   \[ T^n (\alpha U) \cap V \neq \emptyset. \]

3. For each \( x, y \in H \), there are sequences \( \{ x_k \} \) in \( H \), \( \{ n_k \} \) in \( \mathbb{N} \), \( \{ \alpha_k \} \) in \( S \) such that 
   \[ x_k \to x \text{ and } T^n \alpha_k x_k \to y. \]

4. For each \( x, y \in H \), and each neighborhood \( W \) for zero in \( H \), there are \( z \in H \), \( n \in \mathbb{N} \), \( \alpha \in S \) such that 
   \[ x - z \in W \text{ and } T^n \alpha z - y \in W. \]

Proof. 1)\( \Rightarrow \) 2): Let \( \{ W_k \}_{k=1}^\infty \) be a countable base for the topology on \( H \). By \( (2.1) \):
\[ \mathcal{G}_S(T) = \bigcap \left( \bigcup_{\alpha \in S, n \in \mathbb{N}} \left( \frac{1}{\alpha} U_k \right) \right) \text{ for all } k \geq 1, \text{ put } M_k = \bigcup_{n \geq 0} \left[ \frac{1}{\alpha} U_k \right]. \]
By \( (2.2) \)
\[ M_k \text{ is dense for all } k \geq 1. \]
Since \( U \) is open, then there are \( n \in \mathbb{N}, \alpha \in S \) such that 
\[ U \cap T^{-n} \left( \frac{1}{\alpha} V \right) \neq \emptyset \text{ where } V = U \cup W. \]

2)\( \Rightarrow \) 3): Let \( x, y \in H \) for all \( k \geq 1 \), let 
\[ B_k = B \left( x, \frac{1}{\sqrt{k}} \right), \quad B'_k = B \left( y, \frac{1}{\sqrt{k}} \right). \]
By \( (2) \) we get sequences \( \{ n_k \} \) in \( \mathbb{N} \), \( \{ \alpha_k \} \) in \( S \), \( \{ x_k \} \)
in \( H \) such that \( x_k \in B_k \) and \( T^n \alpha_k x_k \in B'_k \) for all \( k \geq 1 \). Then 
\[ \| x_k - x \| < \frac{1}{\sqrt{k}} \] and 
\[ \| T^n \alpha_k x_k - y \| < \frac{1}{\sqrt{k}} \text{ for all } k \geq 1. \]
The desired result follows by letting \( k \to \infty \).

3)\( \Rightarrow \) 4): Let \( x, y \in H \), let \( W \) be a neighborhood for zero in \( H \). By \( (3) \), there are sequences \( \{ x_k \} \) in \( H \), \( \{ n_k \} \) in \( \mathbb{N} \), \( \{ \alpha_k \} \) in \( S \) such that 
\[ x_k \to x \quad \text{and} \quad T^n \alpha_k x_k \to y. \] Hence there is \( k \in \mathbb{N} \) such that 
\[ x_k - x \in W \text{ and } T^n \alpha_k x_k - y \in W. \]
Take \( z = x_k \).

4)\( \Rightarrow \) 3): Let \( x, y \in H \) for all \( k \geq 1 \), let 
\[ B_k = B \left( 0, \frac{1}{\sqrt{k}} \right). \]
By \( (4) \) we get sequences \( \{ z_k \} \) in \( H \), \( \{ n_k \} \) in \( \mathbb{N} \), \( \{ \alpha_k \} \) in \( S \) such that


2.5. Proposition. The operator $T \in \mathcal{G}C_S(H)$ if and only if the set

$$\{x, \alpha T^{n}x \mid x \in H, n \geq 0, \alpha \in S\}$$

is dense in $H \oplus H$.

**Proof.** (i) Let $(y, z) \in H \oplus H$, and let $\epsilon > 0$. Since $T \in \mathcal{G}C_S(H)$, then by (2.3) there are $w \in H, n \geq 0, \alpha \in S$ such that

$$\|w - z\| < \epsilon/2 \quad \text{and} \quad \|\alpha T^n w - z\| < \epsilon/2.$$

Hence

$$\| (w, \alpha T^n w) - (y, z) \|^2 = \|w - y\|^2 + \|\alpha T^n w - z\|^2 < \epsilon^2.$$

(ii) Let $z, y \in H$, and let $\epsilon > 0$. By the conditions there is $t > 0$ and sequences $(\psi_k)$ in $H$, $(\alpha_k)$ in $S$, and $(\eta_k)$ in $\mathbb{C}$ such that

$$| (z, y) - (w_k, \alpha_k T^n \psi_k) | < \epsilon^2$$

for all $k > t$. Hence $\|z - w_k\| < \epsilon$ and $\|y - \alpha_k T^n \psi_k\| < \epsilon$ for all $k > t$. Then let $k \to \infty$ we get $w_k \to z$ and $\alpha_k T^n \psi_k \to y$. Thus by (2.3) $T \in \mathcal{G}C_S(H)$.

Now we will study a condition that implies $G$-cyclic.

2.6. Proposition. Let $T \in B(H)$, let $U, V$ be a nonempty open sets in $H$, and let $W$ be a neighborhood for zero in $H$. If there are $n \geq 0$, $\alpha \in S$ such that $T^n \alpha U \cap W \neq \emptyset$ and $T^n \alpha W \cap V \neq \emptyset$, then $T \in \mathcal{G}C_S(H)$.

**Proof.** We will verify (2.3). Let $x, y \in H$, for all $k \geq 1$, and let $B_k = B(x, \sqrt[2]{k})$.

$$B_k = B(y, \sqrt[2]{k}).$$

By our assumption, there exist sequences $(\eta_k) \in \mathbb{C}$, $(\alpha_k) \in S$, $(w_k) \in W$ and $(z_k) \in H$ such that $z_k \in B_k$, $T^n \alpha_k z_k \in W$ and $T^n \alpha_k w_k \in B_k$ for all $k \geq 1$. Therefore

$$z_k \to x, \quad T^n \alpha_k z_k \to 0 \quad \text{and} \quad T^n \alpha_k w_k \to y.$$ 

The proof complete by taking $x_k = z_k + w_k$ for

Next we will give another characterization of $G$-cyclic operators.

In this section we discuss the properties of the spectrum of $G$-cyclic operator. It is known [6] that
if \( T \) is supercyclic, then \( T^* \) has at most one eigenvalue, hence have

Proposition. Let \( T \in \mathcal{GC}_S(H) \). Then \( T^* \) has at most one eigenvalue with modulus:

1) Greater than one, if \( S \) is bounded above.

2) Less than one, if \( S \) is bounded below.

Proof.

1) Since \( T \in \mathcal{GC}_S(H) \), then \( T \) is supercyclic, thus by [6] \( \sigma_\alpha(T^*) \) contains at most one non-zero eigenvalue, say \( \lambda \). Hence there is a unit vector \( z \in H \) such that \( T^*z = \lambda z \).

Let \( \alpha \in \mathcal{GC}_S(T) \). It is easy to prove that

\[
\left\{ \left( \langle T^*x,z \rangle, \alpha \right) : n \geq 0, \alpha \in S \right\}
\]

is dense in \( \mathbb{R} \) (a).

Note that for \( n \geq 1 \),

\[
\left| \langle T^*x, z \rangle \right| = \left| \alpha \langle T^*x, z \rangle \right|.
\]

Since \( S \) is bounded above, then \( |\alpha| \leq M \) for some \( M \). Now assume that \( |\alpha| \leq 1 \).

Hence

\[
\left| \langle T^*x, z \rangle \right| < M \left| \langle x, z \rangle \right|,
\]

a contradiction with (a).

7) Similar.

It would be useful to say something about Weyl spectrum, \( \sigma_w(T) \), of (cyclic) operators.

Corollary. Let \( T \in \mathcal{GC}_S(H) \), then Weyl spectrum of \( T \) is the spectrum of \( T \) except possibly one element of modulus:

1) Greater than one, if \( S \) is bounded above.

2) Less than one, if \( S \) is bounded below.

Proof.

1) Since \( \sigma_\alpha(T^*) - \sigma_w(T^*) \subseteq \sigma_\alpha(T^*) \) (b), then by (3.1) either \( \sigma_\alpha(T^*) - \sigma_\alpha(T^*) \) or \( \sigma_w(T^*) - \sigma_\alpha(T^*) \) \( (\lambda); |\lambda| \geq 1 \). Hence either

\[
\sigma_w(T^*) = \sigma_\alpha(T^*) \setminus \{ \lambda \}; \quad |\lambda| \geq 1.
\]

or

\[
\sigma_w(T^*) = \sigma_\alpha(T^*) = \sigma(T^*).
\]

2) Similar.

For supercyclic operator \( T \), it is shown in [6] that \( \sigma(T) \subseteq \sigma_\alpha(n) \) is connected for some \( n > 0 \).

Therefore, if \( T \in \mathcal{GC}_S(H) \), then \( \sigma(T) \) is connected for some \( r > 0 \). A question arises: Is there any restriction on \( r ? \)

Proposition. Let \( T \in \mathcal{GC}_S(H) \). Then \( T \in \mathcal{GC}_S(H) \) if one of the following statements holds:

1) \( S \) is bounded, and \( \sigma(T) \) has a component \( \sigma \) such that \( \sigma \subset B(0,1) \).

2) \( S^* \) is bounded and \( \sigma(T) \) has a component \( \sigma \) such that \( \sigma \subset \{ \lambda: |\lambda| > 1 \} \).

Proof.

1) Assume \( T \in \mathcal{GC}_S(H) \). If \( \sigma(T) \) is connected and \( \sigma(T) \subset B(0,1) \), then

\[
\lim_{x \to 0} \left\| T^{n}x \right\| \to 0 \quad \text{for all } x \in H.
\]

Thus

\[
\sup_{x \in H} \left\| T^{n}x \right\| \to \infty \quad \text{as } n \to \infty,
\]

a contradiction with (1.5). Now if \( \sigma \) is a component of \( \sigma(T) \) such that \( \sigma \subset B(0,1) \), then by Riesz decomposition Theorem \( T - T_1 \in \mathcal{GC}_S(H) \) such that \( \sigma(T_1) = \sigma \). But \( T \in \mathcal{GC}_S(H) \) (1.8), hence by the same argument of the first part of this proof we get a contradiction.

2) By using (3.1) and the same argument of the proof of the first part. \( \Box \)

Corollary. Let \( T \in \mathcal{GC}_S(H) \).

1) \( S \) is bounded, then \( \sigma(T) \cap \pi B \) is connected for all \( r > 1 \).

2) \( S^* \) is bounded, then \( \sigma(T) \cap (B(0,r)) \) is connected for all \( r > 1 \).

Proof.

1) Let \( r \leq 1 \). Assume \( \sigma(T) \cap \pi B \) is not connected. Then there is a closed and open subset \( \sigma \) of \( \sigma(T) \cap \pi B \), hence \( \sigma \subset \bar{B} \), thus

\[
\sigma(T) \cap \pi (B) = \emptyset;
\]

hence \( \sigma \subset \pi (B) \). Since \( r \leq 1 \), then \( \sigma \subset B(0,1) \), a contradiction with (3.3).

2) Similar. \( \Box \)

Let's give a simple application of proposition (3.3).

3.2. Corollary. Let \( S \) be a bounded subsemigroup, then a quasinilpotent operator cannot be \( \mathcal{GC}_S \) over \( S \).

Proof. Let \( T \) be a compact operator which is \( \mathcal{GC}_S \) over \( S \). Since every \( \lambda \in \sigma(T) \); \( \lambda \neq 0 \), is an eigenvalue for \( T \), then \( 0 \neq \lambda \in \sigma_p(T^*). \) But by (3.2) either \( \sigma_p(T^*) \supseteq B(0, \lambda) \) or \( \sigma_p(T^*) = \{ \lambda \} \);
\[ |\lambda| > 1. \] Hence either \( \sigma(T) = \{0\} \) or \( \sigma(T) = \{0, \lambda\} \). If \( \sigma(T) = \{0\} \), then we get a contradiction with (3.4). If \( \sigma(T) = \{0, \lambda\} \), then since \( \lambda \) is an isolated point in \( \sigma(T) \), hence \( \{0\} \) is a component for \( \sigma(T) \), a contradiction with (3.4).

References