ABSTRACT

The idea behind this research is that the symmetric group on n letters or any permutation in $S_n$ is a natural setting in which to build and study metaheuristic method applied permutation problems... In [1] they studied the transition matrices of 1-TSP (the traveling salesman of one agent) under the action of conjugation class, in [14] they generalized this result for m-TSP (the traveling salesman problem of multi agent), in [15] they studied the transition matrix of 1-TSP but under the action of the Dihedral group, in this paper we study and generalized this result for m-TSP.

INTRODUCTION

This research uses group theory as unifying mathematical frame work for the study of metaheuristic method, the metaheuristic methods apply to practical combinatorial optimization problems whose solutions are partitions circular ordered arrangement of letters, such solutions are permutations in the symmetric group.

Group theory the "algebra of permutation" can powerfully enhance the study, understanding and application of metaheuristic search neighborhoods.

For the simplicity and clarity of explanation , we will initially focus our consideration on special case of classical m-TSP where the agent do not share a common base or depot city rather each agent is base on one of cities in the subtour cycle assigned to that agent .

In [3] and [14] show that conjugative move methods based upon conjugacy class to build neighborhood that preserve cycle structure. Here and in [15] we compute for subgroup-conjugative move method both conjugacy classes and subgroups provide useful ways to present move method , but of the two , only subgroups grant access to machinery of group action . Here we study the dihedral groups which are subgroup of $S_n$ and its action on conjugacy class and build its transition matrices, Colletti and Barnes in [ 1] investigate the transition matrices of 1-TSP, and in [14] they generalize these results on m-TSP .

In [15 ] she studied the transition matrix of 1-TSP under the action of dihedral group .
In this research we generalize this result for m-TSP under the action of the Dihedral groups.

§ 1: Group theory perspective
The conjugacy class $C(n)$ of $n$-cycle in $S_n$ (the permutation group on $n$ letters) represent the 1-TSP see [11], for m-TSP we need the following due to J.J. Rotman [13]

Definition 1.1: If $\pi$ is an element of $S_n$, then the ordered length $\alpha_i(\pi) = 1 \leq i \leq c(\pi)$ where $c(\pi)$ is the number of the cyclic factor of $\pi$ in cycle notation form a unique determined partition of $n$ which will be called the cycle portion of $\pi$ and which we denoted by $\alpha(\pi)$ where $\alpha(\pi) := (\alpha_i(\pi), ..., \alpha_i(\mu))$

In this paper we take m-TSP where $m = c(\pi)$ and $X$ the conjugacy class of some $\alpha(\pi)$.

Example 1.1: $\pi=(123)(456)(78)(9\ 10\ 11\ 12)$ an element of $S_{12}$, then $\alpha(\pi)=(3,3,2,4)$ and the number of the cyclic factor $c(\pi) = 4$ this element can be consider as an element of the conjugacy class $X$ whose elements have the form (3,3,2,4,) that is each element of $X$ is: $x = (aaa)(aaa)(aa)(aaaa)$ and this represent 4-TSP.

Note: The number of elements of $X$ (i.e. $|X|$) depend on the cycle structure of its elements that is on $\alpha(\pi)$ as follows: If $\alpha(\pi) = (a_1^{n_1}, ..., a_m^{n_m})$ then $|X| = \frac{|S_n|}{a_1^{n_1}a_2^{n_2}...a_m^{n_m}m!}$

Example 1.2: If we take example 1 and we want to find the number of element of the conjugacy class $X$ we note that $x=(aaa)(aaa)(aa)(aaaa)$ that is $\alpha(x) = (3^2, 2, 4)$ then $|X| = \frac{12!}{3^2.2.1.21.1.4.1!} = 3326400$ different element.

Definition 1.2: If $G = \{x^iy^j \mid i = 0,1 \text{ and } j = 0,1, ..., n-1, x^2 = e = y^n, xy = y^{-1}x\}$.

Then $G$ is a group, called the Dihedral group denoted by $G=D_n$ (n≥3) and the order of $G$ we write $o(G) = 2n$.

Definition 1.3: "Group action
Group action of a group $G$ on a set $T$ denoted by $gT$ is a function from the Cartesian product $T \times G$ into $T$ with the properties below: The assignment of $(s, g) \in T \times G$ to $t \in T$ is denoted here as $s^g = t$ [4]:

- $t^e = t$ (e is the identity)
- $t^{gh} = t^g h$ for all $t \in T$ and $g, h \in G$

The group action partition $T$ into cells called orbits, put another way $x, y \in T$ are in the same orbit iff each can be reached from the other via $G$, i.e. $\exists g \in G$ such that $x^g = y$.
Definition 1.4: "Stabilizers"

Stabilizers arise from a group action \( gT \) is an arbitrary set \( T \) together with a group \( G \). Stabilizers are subgroups of a group \( G \) that either individually fix specific elements of \( G \) or else fix subset of \( G \) (i.e. subset element rearrange among themselves).

**Point stabilizer:** Let \( t \in T \) in the group action \( gT \) the point stabilizer of \( t \) in \( G \) is the set of all elements that fix \( t \) under the group action: 
\[
\text{Stab}(G,t) = \{ g \in G : t^g = t \}
\]
And is a subgroup of \( G \) [Dixon 1973, Gallian 1994], for example if \( G \) is \( S_4 \)

- \( T \) be the transpositions in \( S_4 \) the stabilizer of \( t=(2\ 4) \) is the subgroup:
  \[
  \{i, (1\ 3), (2\ 4), (1\ 3)(2\ 4)\}
  \]
  the set of all \( g \in S_4 \) for which \( (2\ 4)^g = (2\ 4) \)

- \( T \) be the 4-cycle in \( G \) and \( t=(1\ 2\ 3\ 4) \) the stabilizer of \( t \) is
  \[
  \{i, (1\ 3)(2\ 4), (1\ 2\ 3\ 4), (1\ 4\ 3\ 2)\}
  \]
  i.e. the set of all \( g \in S_4 \) for which \( (1\ 2\ 3\ 4)^g = (1\ 2\ 3\ 4) \).

In both examples, \( T \) had the property required of group action the operation must be closed on \( T \) and conjugation preserve cycle structure.

In the following section we will recall the work of Colletti & Barens in [7]

§ 2: Types of transition matrices of 1-TSP under conjugation move strategy

Consider swap moves (all possible arrangement of two cities) on a 5-city 1-TSP traveling salesman problem of one agent. In this case solutions are made up of the conjugacy class of 5-cycles, i.e., single agent tours, where the agent visits all five cities (and they assume the agent is based on one of the cities). In this case there are \( 4! = 24 \) tours indexed in dictionary order as follows:

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(12345)</td>
<td>(12354)</td>
<td>(12435)</td>
<td>(12453)</td>
<td>(12534)</td>
<td>(12543)</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>(13245)</td>
<td>(13254)</td>
<td>(13425)</td>
<td>(13452)</td>
<td>(13524)</td>
<td>(13542)</td>
</tr>
<tr>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
</tr>
<tr>
<td>(14235)</td>
<td>(14253)</td>
<td>(14325)</td>
<td>(14352)</td>
<td>(14523)</td>
<td>(14532)</td>
</tr>
<tr>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
</tr>
<tr>
<td>(15234)</td>
<td>(15243)</td>
<td>(15324)</td>
<td>(15342)</td>
<td>(15423)</td>
<td>(15432)</td>
</tr>
</tbody>
</table>

Each of the 24 tours has ten one step neighbors, as pictured in the schematic transition matrix given in Figure (1). However, by using group theory to find the orbits of a group action of the alternating group
, $A_n$, acting upon all 24 tours we can use the orbits to partition the tours into two essential classes of the same size. Identically permuting the rows and columns of the transition matrix in accordance with the partition (with some minor reordering) yields the periodic transition matrix of order 2, presented in Figure (2).

![Fig-1: The “raw” swap neighborhood for $S_5$.](image1)

![Fig-2: The “Revealed” swap neighborhood for $S_5$ (the row and column label has no mean in this figure)](image2)

This means that a swap move neighborhood on a 5-city TSP “communicate”, i.e., we cannot get to any other tour starting at an
arbitrary tour. However in one step you can move only from one essential class to the other.

Figure (3) presents the transition matrix, viewed through the perspective of group theory, for the rearrangement neighborhood of order 3, i.e., all possible 3-city rearrangement.

This non-intuitive result shows that the 3-city rearrangement neighborhood does not communicate! You can never depart the essential class in which you started. Half the tours are unreachable from any specific tours.

If n is an odd number, this surprising structure is present for any n-city 1-TSP when the 3-city rearrangement neighborhood is used! If you are starting solution is not in the same essential class with an optimal tour, you cannot find an optimal solution. However, we may view this structure from a more favorable perspective. If we are aware of this structure, the search for the optimal may be attacked as two independent subproblems with communicating solution neighborhood of dimensionality \( \frac{(n-1)!}{2} \) by \( \frac{(n-1)!}{2} \). For larger values of n, this is considerably less formidable than the original solution neighborhood with dimension \( (n-1)! \) by \( (n-1)! \).

Using group theory Colletti, Barnes and Neuway [7] prove that for any n-city 1-TSP, if you are using conjugative rearrangement moves with a single communicating essential class, or 2 essential classes. If you
have 2 classes they will either communicate with period two or they form two non-communicating classes. They also derive the detailed, specific conditions on the defining conjugacy class and the values of n that lead to each of these structures.

**Definition 2.1:**

For simplicity and clarity, we define the following notations:

- **X** = The set of tours (non-trivial conjugacy class in $S_n$), i.e., all single or multiple tours for a k-city TSP where $k \leq n$.
- **C** = The conjugacy move strategy (a non-trivial conjugacy class of $S_n$).
- **$P^C$** = the 1-move neighborhood of $p \in X$ under the move strategy $C$.
- A conjugative neighborhoods $|X| \times |X|$ transition matrix $T$ has binary entries. The entry corresponding to $p, q \in X$ is:

$$T_{pq} = \begin{cases} * & \text{if } \ldots, p' = q, \ldots, \text{for some } c \in C \\ 0 & \text{otherwise} \end{cases}$$

This paper shows that the transition matrix is diagonalizable into one of the following three types (the $\wedge$ indicates a block containing some non-zero entries):

1. $A_n X$ consists of one orbit and the transition matrix has the form

   $\begin{array}{c|c} \text{orbit (1)} \\ \hline \wedge \end{array}$

2. $A_n X$ has two communicating orbits in alternating moves with period two. Any neighboring pair of elements of $X$ resides in different orbits. The transition matrix has the form:

   $\begin{array}{c|cc} \text{orbit(1)} & \text{orbit(2)} \\ \hline \text{orbit(1)} & 0 & \wedge \\ \wedge & 0 \end{array}$

3. $A_n X$ has two non-communicating orbits. That is any neighboring pair of elements of $X$ is co-orbital, the transition matrix has the form:

   $\begin{array}{c|cc} \text{orbit(1)} & \text{orbit(2)} \\ \hline \text{orbit(1)} & \wedge & 0 \\ 0 & \wedge \end{array}$

At this point this result is a general for 1-TSP or m-TSP, i.e., all transition matrix has one of the previous types, but a natural question is "which type of transition matrix is implied by a given $X$ and $C$?"
When X consists of the set of all n-cycles, the answer depends on two factors W. Barnes et al. [7].

- Does the conjugacy class of n-cycles, X consist of even or odd permutation (that is, is n odd or even)?
- Does C consist of even or odd permutation?
- If n is even, < p > contains the odd permutation, whenever q ∈ S_n is odd, the right coset < p >q has the even element pq whenever q is even, < p >q has an even element (q itself since the identity element belongs to all subgroup and in particular < p >).

Thus < p >q always has an even permutation and so as stated above, the group action only has one orbit.

- If n odd then all elements in < p > are even, if q is odd then < p >q only has odd elements, otherwise < p >q is entirely even. Thus the group action has two orbits.

In summary, for n-cycles in S_n, if n is odd, then there are two orbits, if n is even, there is one orbit.

If there are two orbits, will the transition matrix be of type 2 or type (3)?

This is determined entirely by C. For any q ∈ C, if C has even cycle structure, then A_n q = A_n. Suppose we select, from the same q, p ∈ X (without loss of generality, let q ∈ p^{A_n}) since the 1-move neighborhood of q is q^C, this implies that the union of all orbital element neighborhoods is (p^{A_n})^C = p^{A_n} C. So if all members of C are even permutations then:

p^{A_n} C = p^{A_n} and so this union is the orbit itself see fig. (3). If C consists of odd permutations then p^{A_n} C = p^{S_n \setminus A_n} and so the union is the complementary orbit that does not contain p see figure (2), see [7].

Therefore, if C has an even cycle structure, the transition matrix is of type (3) (two non communicating sets). When C has odd cycle structure, the transition matrix is of type (2) (cycle with period 2).

Note: The two orbits of A_5 on X are under transposition:

(12345)^A_5 = \{ 1, 4, 5, 8, 9, 12, 13, 16, 17, 20, 21, 24 \}.
(12354)^A_5 = \{ 2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23 \}.

These orbits agree with the partitions found in figure 2 and 3, the orbit’s elements are reordered in figure 2 and 3 to place the 5-cycle permutation inverses adjacent and to maximally reveal the matrix structure.

Perhaps the most interesting of these types is type (2) where there is no way to move from one essential class to another. This can be present a
Transition Matrices of Multiple Traveling Salesman Problem under the Action of The Dihedral Group

Arbah and Entisar

trap for individuals using such neighborhood who are not aware of the existence of such structure.

If the search starts in an orbit where no optimal solution resides the search is a priori doomed to failure. On the other hand, persons aware of such structure can make use of it.

§3: Transition matrices of m-TSP under the action of Dihedral groups:

Now we will generalize the results of [15] which was the explanation of 1-tsp under dihedral group now we work with m-tsp:

In the following we will show that the transition matrix of m-TSP under the dihedral group also will be of type two not exactly since we will see that the matrices have more than two orbits but all of them are non communicating orbits. We will start by this example:

Example 3.1: Consider in the symmetric group $S_5$ the conjugacy class $X=(2,3)$ i.e. the cycles of $X$, have the form (xx)(xxx) it is 2-TSP then by the note after definition 1 we get $|X| = \frac{|S_5|}{a_1^{n_1}n_1!...a_m^{n_m}n_m!} = \frac{5!}{2.3} = 20$

different elements all indexed in dictionary order as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1 2)(3 4 5)</td>
<td>(1 2)(3 5 4)</td>
<td>(1 3)(2 4 5)</td>
<td>(1 3)(2 5 4)</td>
<td>(1 4)(2 3 5)</td>
</tr>
<tr>
<td>6</td>
<td>(1 4)(2 5 3)</td>
<td>(1 5)(2 3 4)</td>
<td>(1 5)(1 4 3)</td>
<td>(2 3)(1 4 5)</td>
<td>(2 3)(1 5 4)</td>
</tr>
<tr>
<td>11</td>
<td>(2 4)(1 3 5)</td>
<td>(2 4)(1 5 3)</td>
<td>(2 5)(1 3 4)</td>
<td>(2 5)(1 4 3)</td>
<td>(3 4)(1 2 5)</td>
</tr>
<tr>
<td>16</td>
<td>(3 4)(1 5 2)</td>
<td>(3 5)(1 2 4)</td>
<td>(3 5)(1 4 2)</td>
<td>(4 5)(1 2 3)</td>
<td>(4 5)(1 3 2)</td>
</tr>
</tbody>
</table>

We will discuss the transition matrices of $X$ under the action (move strategy $X^{D_n} = \{x^d : x \in X, d \in D_n\}$ ) of different $D_n := (D_3, D_4, D_5)$:

Note: the row and column label has no mean in the figures below.

1. Case one: $D_3 = \{ i, (1 2 3), (1 3 2), (1 2), (1 3), (2 3) \}$ is the
move strategy then we will get five non communicated orbits which are:

- \((1 2)(3 4 5)^{D_3} = 1^{D_3} = \{(1 2)(3 4 5), (2 3)(1 4 5), (1 3)(2 4 5)\} = \{1, 9, 3\}\)
- \((1 2)(3 5 4)^{D_3} = 2^{D_3} = \{(1 2)(3 5 4), (2 3)(1 5 4), (1 3)(2 5 4)\} = \{2, 4, 10\}\)
- \((1 4)(2 3 5)^{D_3} = 5^{D_3} = \{(1 4)(2 3 5), (2 4)(1 5 3), (3 4)(1 2 5), (2 4)(1 3 5), (1 4)(2 5 3), (3 4)(1 5 2)\} = \{5, 12, 15, 11, 6, 16\}\)
- \((1 5)(2 3 4)^{D_3} = 7^{D_3} = \{(1 5)(2 3 4), (2 5)(1 4 3), (3 5)(1 2 4), (2 5)(1 3 4), (1 5)(2 4 3), (3 5)(1 4 2)\} = \{7, 14, 17, 13, 8, 18\}\)
- \((4 5)(1 2 3)^{D_3} = 19^{D_3} = \{(4 5)(1 2 3), (4 5)(1 3 2)\} = \{19, 20\}\)

Fig-4: the transition matrix of 2-TSP under the action \(D_3\)

2. **Case two:** \(D_4 = \{i, (1 2 3 4), (1 3)(2 4), (1 4 3 2), (1 4)(2 3), (1 2)(3 4), (2 4), (1 3)\}\) is the move strategy then we will get three non communicated orbits which are:

- \((1 2)(3 4 5)^{D_4} = 1^{D_4} = \{(1 2)(3 4 5), (2 3)(1 5 4), (3 4)(1 2 5), (1 4)(2 3 5), (3 4)(1 5 2), (1 2)(3 5 4), (1 4)(2 5 3), (2 3)(1 4 5)\} = \{1, 10, 15, 5, 16, 2, 6, 9\}\)
- \((1 3)(2 4 5)^{D_4} = 3^{D_4} = \{(1 3)(2 4 5), (1 3)(2 5 4), (2 4)(1 3 5), (2 4)(1 5 3)\} = \{3, 4, 11, 12\}\)
- \((1 5)(2 3 4)^{D_4} = 7^{D_4} = \{(1 5)(2 3 4), (2 5)(1 3 4), (3 5)(1 2 4), (4 5)(1 2 3), (4 5)(1 3 2), (2 5)(1 4 3), (1 5)(2 4 3), (3 5)(1 4 2)\} = \{1, 9, 3\}\)
}\} = \{7, 8, 13, 17, 19, 20, 14, 18\}.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{transition_matrix}
\caption{The transition matrix of 2-TSP under the action of $D_4$.}
\end{figure}

3. **Case three**: $D_5 = \{i, (1 2 3 4 5), (1 3 5 2 4), (1 4 2 5 3), (1 5 4 3 2), (1 2)(3 5), (1 3)(4 5), (1 4)(2 3), (1 5)(2 4), (2 5)(3 4)\}$ is the move strategy then we will get two non communicated orbits which are:

- $\{(1 2)(3 4 5)^{D_5} = 1^{D_5} = \{(1 2)(3 4 5), (1 2)(3 5 4), (1 5)(2 3 4), (1 5)(2 4 3), (2 3)(1 4 5), (2 3)(1 5 4), (3 4)(1 2 5), (3 4)(1 5 2), (4 5)(1 2 3), (4 5)(1 3 2) = \{1, 2, 7, 8, 9, 10, 15, 16, 19, 20\}\}$
- $\{(1 3)(2 4 5)^{D_5} = 3^{D_5} = \{(1 3)(2 4 5), (1 3)(2 5 4), (1 4)(2 3 5), (1 4)(2 5 3), (2 4)(1 3 5), (2 4)(1 5 3), (2 5)(1 3 4), (2 5)(1 4 3), (3 5)(1 2 4), (3 5)(1 4 2) = \{3, 4, 5, 6, 11, 12, 13, 14, 17, 18\}\}$. 


In general: Suppose we're given a group action \( H \Omega \) where \( H \) is a subgroup of \( G \) and we want to build an \( \Omega \) by \( \Omega \) transition matrix on \( H \), for \( x, y \in \Omega \) the matrix entry in the "xth" raw and "yth" column gives how many \( h \in H \) satisfy \( x^h = y \). If we arrange \( \Omega \) so that orbits follow one another, it turns out that the matrix block diagonal in which a block's entries are (the same size of any orbital element's stabilizer's).

We now show this although the matrix clearly block diagonal: if \( x, y \in \Omega \) do not share an orbit, then no \( h \in H \) satisfy \( x^h = y \) and so the \((x,y)\)-entry is zero the key is to use the following theorem (in which the group action operator is the conjugation).

**Theorem 3.1:** [Dixon and Mortimer 1996]

Given the group action \( H \Omega \), \( w \in \Omega \) and \( a, b \in H \), \( w^a = w^b \) iff \( a \) and \( b \) are in the same right coset of \( \text{Stab}(H,w) \).

That is if \( y \in \text{Orbit}(H,w) \) then the size of \( \{ h \in H: w^h = y \} \) is that \( \text{Stab}(H,w) \).

Now for our search \( D_n X \) is the group action and for each \( x \in X \) then \( x = p_1 p_2 \ldots p_t \) where \( p_i = (a_1 a_2 \ldots a_m) \), \( i \) is the number of element in each cycle \( p_i \) in \( x \) which depend on the cycle structure of \( x \) then the size of any block in the matrix will be the number of elements of the orbit of an \( x \) equal to:

\[
\text{size of block } 1 = |\text{orbit of } x| = \frac{|D_n|}{|\text{stab}(D_n,x)|}
\]

since the action of each element of \( D_n \) will give different element \( x^{d_i} = x_i \ \forall i = 1,2,\ldots n \) and each element of the stabilizer will give
the same action as in (theorem 1) \( x^a = x^b \). This means:

- The action of each element of \( D_n \) on an element \( x \) will yield different elements that is \( x^{d_i} = y_i \), the order of the orbit of each element of \( X \) will be the same of the order of \( D_n : |\text{orbit of } x| = |D_n| \) this case happened if the stabilizer of \( x \) is \( \{i\} \) only (each of the component of \( x \) not belong to \( D_n \)) but if one of the component of \( x \) belong to \( D_n \) then this component will belong to the stabilizer so the \( |\text{orbit of } x| \) will shrink and will be \( |\text{orbit of } x| = \frac{|D_n|}{|\text{stab}(D_n,x)|} \).

- For each \( y \in \text{orbit } (x) \) we will have \( \text{orbit } (x) = \text{orbit}(y) \) since the action under \( D_n \) is a group action so maps arcs onto themselves via \( D_n \), so if we take \( x \) as a representation element of its orbit we can see the size of block(1) in the transition matrix:  
  \[ \text{size of block } (1) = |\text{orbit of } x| \]
  
- Size of the transition matrix = \( \sum_i |\text{orbit } x_i| \) where the summation is taken among \( x_i \) belong to different orbits.

This prove the following theorems:

**Theorem 3.2:**

Let \( X \) be \( m \)-TSP then the action of \( D_n \) on \( X \) under conjugation action will fall into two cases as follow, for each \( x \in X \) we have:

1. If no one of the component of \( x \in D_n \) then \( |\text{orbit of } x| = |D_n| \)
2. If one of the component of \( x \in D_n \) then

\[ |\text{orbit of } x| = \frac{|D_n|}{|\text{stab}(D_n,x)|} \].

**Theorem 3.3:**
The transition matrix will consist of non-communicating orbits and the size of the transition matrix = \( \sum_i |\text{orbit } x_i| \) where the summation is taken among \( x_i \) belong to different orbits.

**Appendix:**

We use the following program in MATLAB to draw the transition matrices:

```matlab
clear
clc
close all
DIMENSION=input('size of matrix=');
If length(DIMENSION)>1
    ROW=DIMENSION(1);
```
COL=DIMENSION(2);
else
ROW=DIMENSION;
COL=DIMENSION;
end
x (1:ROW,1:COL)=nan;
For i=1:ROW
    Orbit=input([' orbit' num2str(i) ' =     ']);
    x(i , orbit)=1;
end
x1=[1:COL];
figure
hold on
for i=1:ROW
    y=i*x(i, :);
    plot(x1,y,'*')
end
hold off
set(get,'(gcf,'currentaxes'),'XAxisLocation','top')
set(get,'(gcf,'currentaxes'),'Xtick',[1:COL];
set(get,'(gcf,'currentaxes'),'Ytick',[1:ROW];

REFERENCE
6. Colletti, Bruce and J.W. Barnes. _Local Search Structure in the Symmetric Traveling Salesperson Problem under a General Class of Rearrangement Neighborhoods, Applied Mathematics Letters ,
14/1, 105-108 (2001) 


