The Quasi-Controllability for Control Problems in Infinite Dimensional Spaces

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Abstract

In this paper, theorems those deal with the sufficient conditions for the quasi-controllability of the mild solution to the semilinear initial-value control problems in appropriate quasi-Banach spaces are introduced and proved, by using strongly continuous semigroup theory and some techniques of nonlinear functional analysis, such as, fixed point theorem and quasi-Banach contraction principle theorem.

1. Introduction

Control theory in infinite-dimensional spaces is a relatively new field and started blooming only after a well-developed semigroup theory was at hand. Therefore this subject has been interested by many authors. Joshi and Ganesh [1] presented the sufficient conditions which are guarantee the existence of optimal control for some nonlinear control system in Banach space by using semigroup approach. The controllability of semilinear system by employing tools of monotone operator theory and contraction mapping principle introduced in [2]. Al-Moosawy [3] discussed the controllability and optimality of the mild solution for some semilinear initial and boundary control problems in arbitrary Banach spaces, by using semigroup theory and Banach contraction principle theorem. From all the above we find a reasonable justification to accomplish the study of this paper.

Every Banach space is quasi-Banach space, but the converse is not true (see, note 2.1 and Th.2.1). Thus the aim of this paper is to prove that the semilinear initial-value optimal control problems in suitable quasi-Banach spaces are quasi-controllable by using semigroup theory and quasi-Banach contraction principle theorem.

2. Definitions and Theorems

This section contains some definitions and theorems that will be used in the sequel.

Definition 2.1[4]. Let $0 < p < \infty$. Then the collection of all measurable function $f$ for which $|f|^p$ is integrable will be denoted by $L_p(\mu)$. For
each \( f \in L_p(\mu) \), let \( \| f \|_p = (\int |f|^p d\mu)^{1/p} \). The number \( \| f \|_p \) is called the \( L_p \)-norm of \( f \).

**Note 2.1 [5, Note 2.1 and Remark 2.2]**. The space \( L_p(\mu) \) for \( 0 < p < 1 \), is a vector space, but not a normed space (thus not a Banach space).

**Definition 2.2 [5,6]**. A real-valued function \( q \| \cdot \| \) defined on a vector space \( V \) over a field \( F \) is called a quasi-norm if it satisfies the following properties:

1. \( q \| x \| \geq 0 \quad \forall x \in V \) and \( q \| x \| = 0 \iff x = 0 \);
2. \( q \| \alpha x \| = |\alpha| q \| x \| \quad \forall x \in V, \alpha \in F \);
3. There exists a constant \( c \geq 1 \) such that \( q \| x + y \| \leq c (q \| x \| + q \| y \|) \quad \forall x, y \in V \).

The pair \( (V, q \| \cdot \|) \) is called a quasi-normed space, we say simply that \( V \) is quasi-normed space.

**Definition 2.3**. For \( f \in L_p(\mu), 0 < p < 1 \), let us define the quasi-norm of \( f \) which is denoted by \( q \| f \| \) as follows: \( q \| f \| = \| f \|_p = \int |f|^p d\mu \), where \( \| f \|_p \) as defined in def. 2.1.

**Definition 2.4**. Let \( (V, q \| \cdot \|) \) be a quasi-normed space, then

(a) A sequence \( \{x_n\} \) in \( V \) is called convergent to the limit \( x \in V \) if, for \( \varepsilon > 0 \), there exists a positive integer \( N(\varepsilon) \) such that \( q \| x_n - x \| < \varepsilon \quad \forall n \geq N \) (or \( q \| x_n - x \| \to 0 \) as \( n \to \infty \)); (b) A sequence \( \{x_n\} \) in \( V \) is called a Cauchy sequence if, for \( \varepsilon > 0, \exists N(\varepsilon) > 0 \) such that \( q \| x_m - x_n \| < \varepsilon \quad \forall n, m \geq N \) (or \( q \| x_m - x_n \| \to 0 \) as \( n, m \to \infty \)); (c) \( V \) is called a complete quasi-normed space (or quasi-Banach space) if every Cauchy sequence in \( V \) is convergent.

**Theorem 2.1[5, Th.2.6]**. The space \( L_p(\mu) \) for \( 0 < p < 1 \), with the quasi-norm given in def. 2.3 is a quasi-Banach space.

**Definition 2.5**. Let \( T \) be a mapping of a quasi-normed space \( X \) into itself, then \( T \) is called a quasi-contraction mapping if there exists a constant \( k, 0 \leq k < 1 \) such that \( q \| T(x) - T(y) \| \leq k q \| x - y \| \quad \forall x, y \in X \).

**Remark 2.1**. It is clear from the above definition that every quasi-contraction mapping is uniformly continuous.
Theorem 2.2. Every quasi-contraction mapping $T$ defined on a quasi-Banach space $X = L_p(\mu)$ for $0 < p < 1$, into itself has a unique fixed point $x^* \in X$. Moreover, if $x_0$ is any point in $X$ and the sequence $\{x_n\}$ is defined by
\[
x_1 = T(x_0), x_2 = T(x_1), \ldots, x_n = T(x_{n-1}), \ldots
\]
then $\lim_{n \to \infty} x_n = x^*$ and
\[
q\|x_n - x^*\| \leq (ck^n / 1 - k)q\|x_1 - x_0\|, \text{where } c \geq 1 \text{ is a constant (2.1)}
\]

Proof. (Here we use the same approach that used in Banach contraction principle theorem [7] when $X$ is a quasi-Banach space not Banach space).

1. Existence of a fixed point: Let $x_0$ be an arbitrary point in $X$, and we define $x_1 = T(x_0), x_2 = T(x_1), \ldots, x_n = T(x_{n-1}), \ldots$ Then $x_n = T^n(x_0)$ for $n = 1, 2, \ldots$

If $m > n$, say $m = n + d$, $d = 1, 2, \ldots$, then,
\[
q\|x_{n+d} - x_n\| = q\|T^{n+d}(x_0) - T^n(x_0)\| = q\|T(T^{n+d-1}(x_0)) - T(T^{n-1}(x_0))\| \\
\leq k q\|T^{n+d-1}(x_0) - T^{n-1}(x_0)\| \quad [T \text{ is a quasi-contraction mapping}].
\]
Continuing this process $n - 1$ times, we have
\[
q\|x_{n+d} - x_n\| \leq k^n q\|T^d(x_0) - x_0\| \text{ for } n = 0, 1, 2, \ldots, \text{ and all } d. (2.2)
\]
However,
\[
q\|T^d x_0 - x_0\| = q\|T^d x_0 - T^{d-1}x_0 + T^{d-1}x_0 - T^{d-2}x_0 + T^{d-2}x_0 + \cdots + T x_0 - x_0\| \\
\leq c(q\|T^{d-1}x_0 - T^{d-2}x_0\| + q\|T^{d-2}x_0 - T^{d-3}x_0\| + \cdots + q\|T x_0 - x_0\|),
\]
where $c \geq 1$ is a constant. Since $T^d x_0 = T^{d-1}x_1, T^{d-1} = T^{d-2}x_2, \ldots, T x_0 = x_1$, then:
\[
q\|T^d x_0 - x_0\| \leq c(q\|T^{d-1}x_1 - T^{d-2}x_0\| + q\|T^{d-2}x_1 - T^{d-3}x_0\| + \cdots + q\|x_1 - x_0\|).
\]
By equation (2.2) we see that
\[
q\|x_{n+d} - x_n\| \leq c k^n [k^{d-1} q\|x_1 - x_0\| + k^{d-2} q\|x_1 - x_0\| + \cdots + q\|x_1 - x_0\|] \\
\leq c k^n q\|x_1 - x_0\| [1 + k + k^2 + \ldots + k^{d-1} + k^d + \cdots] \text{ as } k \geq 0
\]
or
\[
q\|x_{n+d} - x_n\| \leq c k^n q\|x_1 - x_0\|(1/1 - k), (2.3)
\]
as $\sum_{m=0}^{\infty} k^n$ is a geometric series with common ratio $k$, $0 \leq k < 1$, and the first term 1.
As \( n, m = n + d \to \infty \), from (2.3), we see that \( d_n x_{n+1} - d_n x_n \to 0 \), i.e., \( \{ x_n \} \) is a Cauchy sequence in the quasi-Banach space \( X \) (see, def. 2.4(b)). Thus by Theorem 2.1 and definition 2.4(c), we get that \( \{ x_n \} \) must be convergent say \( \lim_{n \to \infty} x_n = x^* \). Since \( T \) is uniformly continuous, in view of remark 2.1, we have \( T x^* = T(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} \) [by the definition of \( \{ x_n \} \), \( x_{n+1} = T(x_n) \)]
or \( T x^* = x^* \) [the limit of \( \{ x_{n+1} \} \) is the same of \( \{ x_n \} \)]. Thus, \( x^* \) is a fixed point of \( T \).

2. Uniqueness of the fixed point.
Let \( T x^* = x^* \) and \( T x^* = x^* \) hold, then the inequality
\[
\| x^* - x^* \| \leq k \| T x^* - T x^* \| \leq k \| x^* - x^* \|
\]
where the limit exists and the domain of \( A \) is \( D(A) \) exists.

Definition 2.6. A family \( T(t), 0 \leq t < \infty \) of bounded linear operators on a quasi-Banach space \( X \) is called a (one-parameter) semigroup on \( X \) if it satisfies the following conditions:
1. \( T(0) = I \) (I is the identity operator on \( X \))
2. \( T(t + s) = T(t) T(s) \), for each \( t, s \geq 0 \).

Definition 2.7. The infinitesimal generator \( A \) of the semigroup \( T(t) \) on a quasi-Banach space \( X \) is defined by \( A x = \lim_{t \to 0} (1/t)(T(t)x - x) \), where the limit exists and the domain of \( A \) is \( D(A) = \{ x \in X : \lim_{t \to 0^+} (1/t)(T(t)x - x) \} \)

Definition 2.8. A semigroup \( T(t), 0 \leq t < \infty \) of bounded linear operator on a quasi-Banach space \( X \) is said to be strongly continuous semigroup (or \( C_0 \)-semigroup) if:
\[
\| T(t)x_n - x \| \to 0 \quad \text{as} \quad t \to 0^+ \quad \text{for all} \quad x \in X .
\]
For more details about semigroup and \( C_0 \)-semigroup on a Banach space see [8,9].

3. Quasi-Controllability

In this section we will study the existence theorems of the quasi-controllability of the mild solution to the semilinear initial-value control problems in appropriate quasi-Banach spaces, by using semigroup theory and quasi-Banach contraction principle theorem.

3.1 Problem Formulation (1).
Let $X = L_p(\mu)$ for $0 < p < 1$ be a real quasi-Banach space with quasi-norm defined in definition 2.3, and $U$ be a real quasi-Banach space with quasi-norm $\|\cdot\|_q$. And we consider the semilinear optimal control problem in infinite dimensional state space:

$$\dot{x}(t) = Ax(t) + Bu(t) + H(t, N(t, x(t))) \text{ a.e in } J = [0, b], \text{ with } x(0) = x_0.$$  

(3.1)

Where the linear operator $A : D(A) \subset X \rightarrow X$ is the infinitesimal generator of a $C_0$-semigroup defined by $T(t), t \geq 0$, and there exists a constant $M > 0$ such that $\|T(t)\| \leq M$. $B : U \rightarrow X$ is a linear bounded operator, $\|B\| \leq c$, where $c$ is a constant, and the control function $u(.) \in L_p(J, U), 0 < p < 1$ a quasi-Banach space of admissible control functions. Also, let $B_r = \{x \in X; \|x\| \leq r \text{ for some } r > 0\}$.

In order that problem (3.1) makes sense, throughout the paper we assume the following basis hypothesis:

(a) The nonlinear operator $N : J \times X \rightarrow X$, satisfies Lipschitz condition on the second argument $\|N(t, x_1) - N(t, x_2)\| \leq M_2 \|x_1 - x_2\|$, where $M_2$ is a constant and $x_1, x_2 \in B_r$.

(b) The nonlinear operator $H : J \times X \rightarrow X$ is continuous and there exists a constants $M_1, M_3$ such that for all $x_1, x_2 \in B_r$ we have

$$\|H(t, N(t, x_1)) - H(t, N(t, x_2))\| \leq M_1 \|N(t, x_1) - N(t, x_2)\|$$

$$\leq M_1M_2 \|x_1 - x_2\|,$$

and $M_3 = \max_{t \in J} \|H(t, N(t, 0))\|$.

### 3.2 Quasi-controllability Result of Problem (3.1).

It should be noted that to define and find the mild solution to problem (3.1). Let $x(\cdot) \in X$ be the solution of problem (3.1). Since $T(t), t \geq 0$ is the $C_0$-semigroup generated by the linear operator $A$, then by [3, remark (1, 11)] we have that $T(t) x$ is differentiable. That implies the $X$-value function $F(s) = T(t-s)x(s)$ is differentiable for $0 < s < t$, and $(d/ds)F(s) = T(t-s)(d/ds)x(s) + x(s)(d/ds)T(t-s)$, by using (3.1) and [3, Remarks (1.11(5))] we get that

$$(d/ds)F(s) = T(t-s)[Ax(s) + Bu(s) + H(s, N(s, x(s)))] + x(s)[-AT(t-s)]$$

$$= T(t-s)Bu(s) + T(t-s)H(s, N(s, x(s)))$$

(3.2)

Now, by integrating (3.2) from 0 to $t$, we have
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\[ F(t) - F(0) = \int_{0}^{t} T(t-s)Bu(s)ds + \int_{0}^{t} T(t-s)H(s, N(s, x(s)))ds \]

By definition of the semigroup, we have \( T(0) = I \), and by definition of above function \( F(s) \) we get that

\[ x(t) = T(t)x_0 + \int_{0}^{t} T(t-s)Bu(s)ds + \int_{0}^{t} T(t-s)H(s, N(s, x(s)))ds \]

(3.3)

So according to equation (3.3), the following definition has been presented.

**Definition 3.1.** A continuous function \( x(\cdot) \in X \) (real quasi-Banach space) given by equation (3.3) will be called a mild solution to the semilinear initial-value control problem (3.1).

**Definition 3.2.** The system (3.1) is said to be quasi-controllable on the interval \( J = [0, b] \) if for every \( x_0, x_1 \in X \), there exists a control \( u(\cdot) \in L_p(J, U) \) for \( 0 < p < 1 \), such that the mild solution \( x(t) \) defined by (3.3) satisfying \( x(b) = x_1 \).

**Theorem 3.1.** Consider the semilinear optimal control problem (3.1)

\[ \dot{x}(t) = A x(t) + Bu(t) + H(t, N(t, x(t))) \]

a.e in \( J = [0, b] \), with \( x(0) = x_0 \), with hypothesis (a) and (b). Assume further that

(c) Define linear continuous operator \( w \) from \( L_p(J, U) \), \( 0 < p < 1 \), into \( X \) as follows, \( wu = \int_{0}^{b} T(b-s)Bu(s)ds \), and suppose that for every \( u(\cdot) \in L_p(J, U) \), \( 0 < p < 1 \) there exists a constant \( k > 0 \) such that

\[ k \| u \| \leq \| wu \|. \]

**Note 3.1.** From the above condition (c) and [5, Prop. 2.1 and Th. 4.1 (g)] we see that the inverse operator of \( w \) exists and is continuous (bounded). i.e., the operator \( w^{-1} : \text{Rang} w \rightarrow L_p(J, U) \), \( 0 < p < 1 \), defined by \( w^{-1}(wu(t)) = u(t) \) exists and there exists a constant \( k_2 > 0 \) such that

\[ \| w^{-1} \| \leq k_2. \]

(d) \( c_i M_q \| x_0 \| + h_1 + h_2 + b c_i M_k q (\| x_i \| + M_q \| x_0 \| + h_2 (c_i/c_i c_j) + h_3) \leq r \),

where \( h_1 = b c_i c_2 M_M M_2 r \), \( h_2 = b c_i c_2 M_M 3 \), \( h_3 = b c_i M_3 M_2 r \), and \( c_i \geq 1, i = 1, 2, 3 \) are constants.

(e) Let \( d = c_i b M_1 M_2 + c_i b^2 M^2 c_k M_1 M_2 \in [0,1) \) where \( c_i > 1 \) is a constant.
Then the system (3.1) is quasi-controllable on J.

**Proof.** By using definition (3.2) and equation (3.3) we get that

\[ x_1 = x(b) = T(b)x_0 + \int_0^b T(b-s)Bu(s)ds + \int_0^b T(b-s)H(s, N(s, x(s)))ds. \]

Condition (c) leads to

\[ x_1 = T(b)x_0 + wu + \int_0^b T(b-s)H(s, N(s, x(s)))ds. \]

Therefore,

\[ wu = x_1 - T(b)x_0 - \int_0^b T(b-s)H(s, N(s, x(s)))ds. \]

By note 3.1, we have that

\[ u(t) = w^{-1}(wu(t)) = w^{-1}(x_1 - T(b)x_0 - \int_0^b T(b-s)H(s, N(s, x(s)))ds)(t) \]

(3.4) Now, let \( Q = C(J, B_r) \) (the space of all continuous functions defined from J into \( B_r \)), then by using the control \( u(t) \) defined by equation (3.4) and the definition of the mild solution (3.3), we shall show that the operator \( \Phi \) defined by

\[ \Phi x(t) = T(t)x_0 + \int_0^t T(t-s)H(s, N(s, x(s)))ds + \int_0^t T(t-s)Bw^{-1}(x_1 - T(b)x_0 - \int_0^b T(b-s)H(s, N(s, x(s)))ds)(t) \]

(3.5)

has a unique fixed point.

First, we show that \( \Phi \) map \( Q \) into itself. For \( x \in Q \) we have \( B_r = \{ x \in X : q\| x \| \leq r \} \), then \( x(t) \in B_r \), one has to show that \( q\| Qx(t) \| \leq r \) for some \( r > 0 \). From (3.5) and definition (2.2), we get that

\[ q\| \Phi x(t) \| \leq c_1 \left[ \int_0^t q\| T(t) \| q\| x_0 \| + \int_0^t q\| T(t-s) \| q\| H(s, N(s, x(s))) - H(s, N(s, 0)) \right] + H(s, N(s, 0))ds + \int_0^t q\| T(t-s) \| \| B \| \| w^{-1} \| + q\| x_1 \| + q\| T(b) \| q\| x_0 \| + \int_0^b q\| T(b-s) \| q\| H(r_1, N(r_1, x(r_1))) - H(r_1, N(r_1, 0)) + H(r_1, N(r_1, 0)) \| dr_1 \| \| x(t) \| \] \]

where \( c_1 \geq 1 \) be a constant.

Since \( q\| B \| \leq c, \| T(t) \| \leq M \), then by note 3.1, we get that

\[ q\| \Phi x(t) \| \leq c_1 \left[ M q\| x_0 \| + \int_0^t \left( M c_2 \left( q\| H(s, N(s, x(s))) - H(s, N(s, 0)) \right) + q\| H(s, N(s, 0)) \right) ds + \int_0^t M c_2 \left( q\| x_1 \| + M q\| x_0 \| + \int_0^b M c_3 \left( \right) ... \right) \right] \]
\[ q \left\| H(r_t, N(r_t, x(r_t))) - H(r_t, N(r_t, 0)) \right\| + q \left\| H(r_t, N(r_t, 0)) \right\| dr_1(s) ds, \]

where \( c_2 \geq 1 \) and \( c_3 \geq 1 \) be a constants. Condition (b) gives

\[ q \left\| \Phi x(t) \right\| \leq c_1 \left\| M_q x_0 \right\| + \int_0^t M(c_2 (M_1 M_2 q x(s) + M_3)) ds + \int_0^t M(c_3 (M_1 M_2 q x(r_t) + M_3)) dr_1(s) ds \]

Since \( x \in B_r \), then \( q \left\| x \right\| \leq r \) and then

\[ q \left\| \Phi x(t) \right\| \leq c_1 M_q \left\| x_0 \right\| + \int_0^t M(c_2 M_1 M_2 r + bc_{c_2} M_3 + bc_{c_2} M_2 k_2 q x_1) ds + \int_0^t \left[ M(c_3 (M_1 M_2 q x(r_t) + M_3)) dr_1(s) ds \right] \]

By condition (d) we see that

\[ q \left\| \Phi x(t) \right\| \leq c_1 M_q \left\| x_0 \right\| + h_1 + h_2 + bc_{c_2} M_2 k_2 q \left\| x_1 \right\| + M_q \left\| x_0 \right\| + h_2 (c_3 (c_1 c_2)) + h_3 \] \( \leq r \), and this implies that \( \Phi x(t) \in Q \), therefore \( \Phi \) map \( Q \) into itself.

Second, we have to show that \( \Phi \) is a quasi-contraction mapping, for \( x_1(t), x_2(t) \in Q \) and by definition of \( \Phi x(t) \) in (3.5) we get that

\[ q \left\| \Phi x_1(t) - \Phi x_2(t) \right\| = q \left\| T(t) x_0 + \int_0^t T(t-s) H(s, N(s, x_1(s))) ds + \int_0^t T(t-s) H(s, N(s, x_2(s))) ds \right\| B w^{-1} (x_1 - T(b) x_0) - \int_0^t T(t-s) H(r_t, N(r_t, x_1(r_t))) dr_1 ds - T(t)x_0 \]

\[ - \int_0^t T(t-s) H(s, N(s, x_2(s))) ds - \int_0^t T(t-s) B w^{-1} (x_1 - T(b) x_0) \]

\[ - \int_0^t T(t-s) H(r_t, N(r_t, x_2(r_t))) dr_1 ds \]. Then

\[ q \left\| \Phi x_1(t) - \Phi x_2(t) \right\| \leq c_4 \left\| T(t-s) \right\| q \left\| H(s, N(s, x_1(s))) - H(s, N(s, x_2(s))) \right\| ds + \int_0^t q \left\| T(t-s) \right\| \]

\[ q \left\| B \right\| q \left\| w^{-1} \right\| \int_0^t q \left\| H(r_t, N(r_t, x_1(r_t))) - H(r_t, N(r_t, x_2(r_t))) \right\| dr_1 ds \]

where \( c_4 \geq 1 \) is a constant. By conditions (b) and (c), we have that

\[ q \left\| \Phi x_1(t) - \Phi x_2(t) \right\| \leq c_4 \left\| MM_1 M_2 q x_1(s) - x_2(s) \right\| ds + \int_0^t Mck_2 \left\| M MM_1 M_2 q x_1(r_t) - x_2(r_t) \right\| dr_1 ds \]

\[ \leq c_4 b M M_1 M_2 q \left\| x_1(t) - x_2(t) \right\| + c_4 b Mck_2 (b M M_1 M_2) \left\| x_1(t) - x_2(t) \right\| \]

\[ = (c_4 b M M_1 M_2 + c_4 b^2 M^2 c k_2 M_1 M_2) \left\| x_1(t) - x_2(t) \right\| \] [by condition (e)].
Therefore $\Phi$ is a quasi-contraction mapping (see, def.2.5) and hence by theorem (2.2), there exists a unique fixed point $x \in Q$ such that $\Phi x(t) = x(t)$, thus any fixed point of $\Phi$ is a mild solution of (3.1) on $J$, which satisfies $x(b) = x_1$. Hence the system is a quasi-controllable on $J$.

### 3.2 Problem Formulation (2)

Let the quasi-Banach spaces $X$ and $U$ are defined as in problem (3.1), and consider the semilinear optimal control problem in infinite dimensional state space:

$$(d/dt)(x(t) + G(t, x(t))) = A x(t) + B u(t) + H(t, N(t, x(t))) \quad \text{a.e in} J = [0, b],$$

with $x(0) = x_0$.

(3.6) where the linear operators $A$, $B$ and the control function $u(\cdot) \in L_p(J, U), 0 < p < 1$ with $B_r$ are defined as in problem (3.1). We assume with hypothesis (a) and (b) the following condition

(f) The nonlinear operator $G: J \times X \to X$ satisfies Lipschitz condition on the second argument, and for all $x_1, x_2 \in B_r$, $\ell, \ell_2 > 0$, we have:

$$\|g(t, x_1) - g(t, x_2)\| \leq \ell \|x_1 - x_2\| \text{ and } \ell_2 = \max_{t \in J} |g(t, 0)|.$$

**Theorem 3.2.** Consider the semilinear optimal control problem (3.6) with hypothesis (a), (b) and (f). Assume further that

(g) There exists a positive function $f_0 \in L_1(0, b)$ such that

$$\|AT(t)\| \leq f_0(t) \text{ a.e } t \in (0, b).$$

And there exists a constant $k_1 > 0$, such that $\int_0^b f_0(t) dt \leq k_1$.

(h) Define a linear continuous operator $w$ as in (c), with note 3.1.

(m) $c_i M_\ell \|x_1\| + c_i Mh_1 + c_i h_2 (1 + k_i) + b Mcc_i k_2 \|x_1\| + M_\ell \|x_2\| + Mh_1 + \ell \|x_1\| + \ell_2 + h_2 k_i + h_3 c_i h_2 \leq r,$ where $h_1 = \ell \|x_0\| + \ell_2,$

$h_2 = \ell_1 r + \ell_2,$ $h_3 = bM (M_1 M_2 r + M_3)$ and $c_i \geq 1, i = 1, 2$ are constants.

(n) Let $d = c_3 \ell_1 + c_3 \ell k_1 + b Mcc c_2 (\ell, k_1 - bM M_2) + c_3 bM M_1 M_2$ such that $d \in (0, 1)$, where $c_3 \geq 1$ is a constant.

Then the system is a quasi-controllable on $J$.

**Proof.** The proof of theorem 3.2, is similar to the proof of theorem 3.1, where a function $x: [0, b] \to X$ defined by

$$x(t) = T(t) x_0 + T(t) G(0, x) - G(t, x(t)) + \int_0^t T(t - s) B u(s) ds -$$

$$\int_0^t G(s, x(s)) AT(t - s) ds + \int_0^t T(t - s) H(s, N(s, x(s))) ds,$$
is a mild solution to the semilinear initial-value control problem (3.6). And the system (3.6) is a quasi-controllable on $J$ if for every $x_0, x_1 \in X$ there exists a control $u(\cdot) \in L_p(J,U)$, $0 < p < 1$ such that the mild solution satisfy $x(b) = x_1$.

3.3 Application.

The Leslie Model [5] is a powerful tool which uses matrices to determine the growth of a population as well as the age distribution within a population over certain time interval.

**Definition 3.3 [5].** An infinite matrix $(a_{ij})_1^\infty$ whose elements satisfy

$$a_{ij} = \begin{cases} F_i, & i = 1 \text{ and } j = 1,2,\ldots \\ P_i, & i = 2,3,\ldots \text{ and } j = i - 1, \\ 0, & \text{ otherwise}. \end{cases}$$

Where $F_i \geq 0$ is the average reproduction of females in the $i$-th age class, and $0 < P_i < 1$ is the survival rate of a females in the $i$-th age class, is called an infinite dimensional Leslie matrix.

Here we consider a matrix $(a_{ij})_1^\infty$ whose elements are functions in the quasi-Banach space $L_p$ for $0 < p < 1$ (this matrix has positive real eigenvalue and eigenvector corresponding to this eigenvalue [5, Th.4.5]).

Now, define the linear integral equation in $f(s)$ as follows:

$$f(s) = \int_a^b K(s,t) f(t) dt,$$

where $K(s,t)$ is a positive continuous function defined for $a \leq s, t \leq b$, such that $\int_a^b \int_a^b |K(s,t)|^p dsdt < \infty$ for $0 < p < 1$.

Since the elements of a matrix $(a_{ij})_1^\infty$ are functions in the infinite quasi-Banach space $L_p$ for $0 < p < 1$, then $(a_{ij})_1^\infty$ [5] determines a linear operator $T : L_p \to L_p$ defined by $Tf = g$, i.e.,

$$Tf(s) = g(s) = \int_a^b K(s,t) f(t) dt,$$

where

$$T = (a_{ij})_1^\infty.$$

(3.7)

Now, let $X = U = L_p$ for $0 < p < 1$ be a real quasi-Banach spaces, and consider the optimal control problem (3.1), where $A = (a_{ij})_1^\infty$ infinite dimensional Leslie matrix, $B$ is a matrix whose elements are functions in the quasi-Banach space $L_p, 0 < p < 1$, and assume that the operator $H$ in (3.1) is the zero operator.
Then a matrix $A = (a_{ij})^{\infty}_{i,j=1}$ defines a bounded (compact) linear operator $T : L_p \to L_p$ for $0 < p < 1$, where $T$ defined as in (3.7)[5, Th.4.2 and Th.4.3]. By the same way we see that the operator B is bounded. Thus the operator A is the infinitesimal generator of a $C_0$-semigroup defined by $T(t) = e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$, $t \geq 0$, which is bounded [9]. Therefore, it is not difficult to check that all assumptions of theorem 3.1, are satisfied for the above problem.

4. Conclusion

The space $L_p$ is a Banach space for $1 \leq p < \infty$ [4], while it is not a Banach space for $0 < p < 1$ (note 2.1). In this paper we extend the study of controllability for control problems in the spaces of this kind $L_p$, $0 < p < 1$. Thus the notions of a quasi-Banach contraction principle theorems and strongly continuous semigroup is introduced for this space, and used it to prove theorems deals with the quasi-controllability for the control problems in the space $L_p$, $0 < p < 1$.

Future work

The quasi-observability and optimality for the problems (3.1) and (3.6) may be considered.

REFERENCES
