Artin's characters table of the group \((Q_{2m} \times C_2)\) when \(m=2^h\), \(h \in \mathbb{Z}^+\)

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Abstract

The main purpose of this paper is to find the general form of Artin's characters table of the group \((Q_{2m} \times C_2)\) when \(m=2^h\), \(h \in \mathbb{Z}^+\) where \(Q_{2m}\) is the Quaternion group of order \(4m\) and \(C_2\) is the Cyclic group of order 2 this table depends on Artin's characters table of a quaternion group of order \(4m\) when \(m=2^h\), \(h \in \mathbb{Z}^+\). which is denoted by \(\text{Ar}(Q_{2^{h+1}} \times C_2)\).

المستخلص

الهدف الرئيسي لهذا البحث هو إيجاد الصيغة العامة لجدول شواخص أرتين للزمرة \((Q_{2m} \times C_2)\) حيث أن \(Q_{2m}\) هي الزمرة الرباعية العوممية ذات الرتبة 2 حينما \(m=2^h\), \(h \in \mathbb{Z}^+\), الجدول يعتمد على جدول شواخص أرتين للزمرة الرباعية العوممية ذات الرتبة 4m عندما \(m=2^h\), \(h \in \mathbb{Z}^+\). الذي يعبر عنه \(\text{Ar}(Q_{2^{h+1}} \times C_2)\).

Introduction

For a finite group G, let \(R(G)\) denote the group generated by \(z\)-valued characters of the group G. Inside this group, we have a subgroup generated by Artin's characters (the characters induced from the principal characters of cyclic subgroups) of G which will be denoted by \(T(G)\). the factor group \(R(G)/T(G)\) which is denoted by \(A(G)\) is called Artin's cokernel of G characters and it is a finite abelian group of the exponent \(A(G)\) which is called Artin's exponent. Let \(x\) and \(y\) be two elements of \(G\), \(x\) and \(y\) are called \(\Gamma\)-conjugate if the cyclic subgroups which they generate, are \(\Gamma\)-conjugate in \(G\). this is defined as an equivalent relation on \(G\), its classes are called \(\Gamma\)-classes of \(G\).

The square matrix whose rows correspond to Artin's characters and columns correspond to the \(\Gamma\)-classes of \(G\) is called Artin's characters table. this matrix is very important to find the cyclic decomposition of the factor group \(A(G)\) and Artin's exponent \(A(G)\).


The aim of this paper is to find the general form of the Artin's characters table of the group \((Q_{2m} \times C_2)\) when \(m=2^h\), \(h \in \mathbb{Z}^+\).

1. Preliminaries

This section introduce some important definitions and basic concepts of the Artin's characters tables, the Artin's characters table of \(C_{r^*}\), the Artin's characters table of the Quaternion group \(Q_{2m}\) when \(m\) is an even number, the Artin's characters table of the Quaternion group \(Q_{2m}\) when \(m=2^h\), \(h \in \mathbb{Z}^+\) and the Group \((Q_{2m} \times C_2)\).
1.1 Definition: [7]
Two elements of $G$ are said to be $\Gamma$-conjugate if the cyclic subgroups they generate are conjugate in $G$, this defines an equivalence relation on $G$. It is classes are called $\Gamma$-classes.

1.2 Example:
Consider a cyclic group $C_4 = \langle x \rangle$ such that:
1 is $\Gamma$-conjugate 1
Then the $\Gamma$-class $[1] = \{1\}$
$\langle x \rangle = \langle x^3 \rangle$
Then $x$ and $x^3$ are $\Gamma$-conjugate, and $[x] = \{x, x^3\}$
There is another $\Gamma$-class $[x^1] = \{x^2\}$
So that there are three $\Gamma$-classes of $C_4 : [1], [x]$ and $[x^2]$

In general for $C_p$, where $p$ is any prime number, so that are $s+1$ distinct

$\Gamma$-classes which are $[1], [x], [x^p], \ldots, [x^{p^s-1}]$.

1.3 Definition: [5]
Let $H$ be a subgroup of $G$ and let $\phi$ be a class function on $H$, the induced class function on $G$, is given by:

$$
\phi'(g) = \frac{1}{|H|} \sum_{x \in G} \phi^*(xgx^{-1})
$$

where $\phi^*$ is defined by:

$$
\phi^*(h) = \begin{cases} 
\phi(h) & \text{if } h \in H \\
0 & \text{if } h \notin H
\end{cases}
$$

1.4 Proposition: [3]
Let $H$ be a subgroup of $G$ and $\phi$ be a character of $H$, then $\phi'$ is a character of $G$ and it is called induced character on $G$

1.5 Example:
Take $H=C_4$ as acyclic subgroup of $Q_4$ the character $\phi$ on $C_4$ is defined as follows: $\phi(1) = 1, \phi(x) = \omega, \phi(x^2) = \omega^2, \phi(x^3) = \omega^3$

Where $\omega = e^{2\pi i / 4}$

$$
\phi'(1) = \frac{1}{|H|} \sum_{r \in Q_4} \phi^*(r.1.r^{-1}) = \frac{1}{|H|} \sum_{r \in Q_4} \phi(1) = \frac{1}{4}(1+1+1+1+1+1+1+1+1+1) = \frac{1}{4}.8 = 2
$$

$$
\phi'(x) = \frac{1}{|H|} \sum_{r \in Q_4} \phi^*(r.x.r^{-1}) = \frac{1}{|H|} \sum_{r \in Q_4} \phi(x) = \frac{1}{|H|} [\phi(x) + \phi(x) + \phi(x) + \phi(x^3) + \phi(x^3) + \phi(x^3) + \phi(x^3)] = (1/4)4(\phi(x) + \phi(x^3)) = \omega + \omega^3
$$

$$
\phi'(x^2) = \frac{1}{|H|} \sum_{r \in Q_4} \phi^*(r.x^2.r^{-1}) = \frac{1}{|H|} \sum_{r \in Q_4} \phi(x^2) = \frac{1}{|H|} [\phi(x^2) + \phi(x^2) + \phi(x^2) + \phi(x^2) + \phi(x^2) + \phi(x^2) + \phi(x^2) + \phi(x^2)] = (1/4).8 \phi(x^2) = 2\omega^2
$$
\[
\phi'(x^3) = \frac{1}{|H|} \sum_{x \in C_H} \phi'(x^3) \phi(x)
\]

Since \( y, xy, x^2y, x^3y \notin C_4 \) then \( \phi'(y) = \phi'(xy) = \phi'(x^2y) = \phi'(x^3y) = 0 \)

Hence \( \phi' \) is induced characters of \( Q_4 \).

**1.6 Theorem:** If \( H \) be a cyclic subgroup of \( G \) and \( h_1, h_2, \ldots, h_m \) are chosen representative for \( m \)-conjugate classes, then :

1. \( \phi'(g) = \frac{C_g(g)}{C_H(g)} \sum_{i=1}^{m} \phi(h_i) \) if \( h_i \in H \cap \text{CL}(g) \)

2. \( \phi'(g) = 0 \) if \( H \cap \text{CL}(g) = \phi \)

**1.7 Example:**
To find the Artin's character of \( C_4 \), there are three cyclic subgroups of \( C_4 \), which are \( \{1\}, <x> \text{ and } <x^2> \), there are three \( \Gamma \)-classes which are \( [1] = \{1\}, [x^2] = \{x^2\} \) and \( [x] = \{x,x^3\} \)

So we have three distinct Artin's characters, then by using theorem (1.6)

\[
\phi'(g) = \frac{C_g(g)}{C_H(g)} \sum_{i=1}^{m} \phi(h_i) \text{ if } h_i \in H \cap \text{CL}(g)
\]

\( \phi'(g) = 0 \text{ if } H \cap \text{CL}(g) = \phi \)

(i) if \( H = \{1\} \text{ and } G = C_4 \)

since \( H \cap \text{CL}(1) = \{1\} \), then

\( \phi'(1) = \frac{2^1}{1} \cdot \phi(1) = 2^1 \cdot 1 = 2 \)

since \( H \cap \text{CL}(x) = \phi \), then \( \phi'(x) = 0 \)

since \( H \cap \text{CL}(x^2) = \phi \), then \( \phi'(x^2) = 0 \)

(ii) if \( H = <x^2> = \{1,x^2\} \)

\( \phi'(1) = \frac{2^2}{2} \cdot \phi(1) = 2 \cdot 2 = 4 \), since \( H \cap \text{CL}(1) = \{1\} \)

\( \phi'(x^2) = \frac{2^2}{2} \cdot \phi(1) = 2 \cdot 2 = 4 \), since \( H \cap \text{CL}(x^2) = \{x^2\} \)

since \( H \cap \text{CL}(x) = \phi \), then \( \phi'(x) = 0 \)

(iii) if \( H = <x> = \{1,x,x^2,x^3\} \)

\( \phi'(1) = \frac{2^2}{2} \cdot \phi(1) = 2 \cdot 2 = 4 \), since \( H \cap \text{CL}(1) = \{1\} \)

\( \phi'(x^2) = \frac{2^2}{2} \cdot \phi(1) = 2 \cdot 2 = 4 \), since \( H \cap \text{CL}(x^2) = \{x^2\} \)

\( \phi'(x) = \frac{2^2}{2} \cdot \phi(1) = 2 \cdot 2 = 4 \), since \( H \cap \text{CL}(x) = \{x\} \)

Then we get three Artin's characters \( \phi_1', \phi_2' \) and \( \phi_3' \).

**1.8 Definition:**
Let \( G \) be a finite group, all characters of \( G \) induced from a principal character of cyclic subgroups of \( G \) are called **Artin's characters of** \( G \).

In theorem (1.6), if \( \phi \) is the principal character, then \( \phi(h_i) = \phi(1) = 1 \), where \( h_i \in H \)
1.9 Proposition:[2]  
The number of all distinct Artin's characters on a group $G$ is equal to the number of $\Gamma$-classes on $G$. 
Furthermore, Artin's characters are constant on each $\Gamma$-classes.

1.10 Definition: [1]  
Artin’s characters of finite group $G$ can be displayed in table called Artin’s characters table of $G$ which is denoted by $Ar(G)$.

The first row is the $\Gamma$-conjugate classes, the second row is the number of elements in each conjugate classes, the third row is the size of the centralize $|C_G(CL_\alpha)|$ and the rest row contain the values of Artin’s characters.

1.11 Example:  
In the Artin’s character table of $C_4$ there are three $\Gamma$-classes, $[1]$, $[x^2]$ and $[x]$ then, from proposition (1.9) they obtain three distinct Artin's characters 
And From example (1.7) we obtain the values of Artin’s characters, then the table of it as follows:

<table>
<thead>
<tr>
<th>$\Gamma$-classes</th>
<th>$[1]$</th>
<th>$[x^2]$</th>
<th>$[x]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CL_\alpha$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$C_{C_4}(CL_\alpha)$</td>
<td>$2^3$</td>
<td>$2^3$</td>
<td>$2^3$</td>
</tr>
<tr>
<td>$\phi'_1$</td>
<td>$2^2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\phi'_2$</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$\phi'_3$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table (1)

1.12 Theorem:[1]  
The general form of Artin’s character table of $C_{p^s}$ when $p$ is a prime number and $s$ is an integer number is given by:

<table>
<thead>
<tr>
<th>$\Gamma$-classes</th>
<th>$[1]$</th>
<th>$[x^{p^{s-1}}]$</th>
<th>$[x^{p^{s-2}}]$</th>
<th>$[x^{p^{s-3}}]$</th>
<th>...</th>
<th>$[x^p]$</th>
<th>$[x]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CL_\alpha$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$C_{p^s}(CL_\alpha)$</td>
<td>$p^s$</td>
<td>$p^s$</td>
<td>$p^s$</td>
<td>$p^s$</td>
<td>...</td>
<td>$p^s$</td>
<td>$p^s$</td>
</tr>
<tr>
<td>$\phi'_1$</td>
<td>$p^s$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\phi'_2$</td>
<td>$p^{s-1}$</td>
<td>$p^{s-1}$</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\phi'_3$</td>
<td>$p^{s-2}$</td>
<td>$p^{s-2}$</td>
<td>$p^{s-2}$</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\phi'_s$</td>
<td>$P$</td>
<td>$P$</td>
<td>$P$</td>
<td>$P$</td>
<td>...</td>
<td>$P$</td>
<td>$P$</td>
</tr>
<tr>
<td>$\phi'_{s+1}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table (2)
1.13 Example:
Consider the cyclic group $C_{128}$, To find the Artin’s character table we use theorem (1.12) as follows: The group $C_{128} = C_{2^7}$ then $\text{Ar}(C_{2^7}) =$

<table>
<thead>
<tr>
<th>$\Gamma$- classes</th>
<th>$[1]$</th>
<th>$[x^{2^1}]$</th>
<th>$[x^{2^2}]$</th>
<th>$[x^{2^3}]$</th>
<th>$[x^{2^4}]$</th>
<th>$[x^{2^5}]$</th>
<th>$[x^{2^6}]$</th>
<th>$[x]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>CL_\alpha</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$</td>
<td>C_{c_2}(CL_\alpha)</td>
<td>$</td>
<td>$2^7$</td>
<td>$2^7$</td>
<td>$2^7$</td>
<td>$2^7$</td>
<td>$2^7$</td>
<td>$2^7$</td>
</tr>
<tr>
<td>$\phi_i'$</td>
<td>$2^7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\phi_2'$</td>
<td>$2^6$</td>
<td>$2^6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\phi_3'$</td>
<td>$2^5$</td>
<td>$2^5$</td>
<td>$2^5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\phi_4'$</td>
<td>$2^4$</td>
<td>$2^4$</td>
<td>$2^4$</td>
<td>$2^4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\phi_5'$</td>
<td>$2^3$</td>
<td>$2^3$</td>
<td>$2^3$</td>
<td>$2^3$</td>
<td>$2^3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\phi_6'$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\phi_7'$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table (3)

1.14 Theorem: [8]
The Artin’s characters table of the Quaternion group $Q_{2m}$ when $m$ is an even number is given as follows:

<table>
<thead>
<tr>
<th>$\Gamma$- classes</th>
<th>$[1]$</th>
<th>$[x^{m/2}]$</th>
<th>$[x^m]$</th>
<th>$[x]$</th>
<th>$[y]$</th>
<th>$[xy]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>CL_\alpha</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$</td>
<td>C_{c_m}(CL_\alpha)</td>
<td>$</td>
<td>$4m$</td>
<td>$4m$</td>
<td>$2m$</td>
<td>$2m$</td>
</tr>
<tr>
<td>$\Phi_1$</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Phi_2$</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Phi_i$</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Phi_{i+1}$</td>
<td>m</td>
<td>m</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Phi_{i+2}$</td>
<td>m</td>
<td>m</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table (4)

where $l$ is the number of $\Gamma$- classes of $C_{2m}$ and $\Phi_j$ ; $1 \leq j \leq l+2$ are the Artin characters of the Quaternion group $Q_{2m}$.

Let $m=2^h$, $h \in \mathbb{Z}^+$ then $\text{Ar}(Q_{2m})=\text{Ar}(Q_{2^{h+1}})$ and it is given by:
1.15 Example:
To construct $\text{Ar}(Q_{128})$ by using theorem (1.14) we get the following table:

$\text{Ar}(Q_{128}) = \text{Ar}(Q_{72}) = \text{Ar}(C_{2})$

Table (5)

<table>
<thead>
<tr>
<th>$\Gamma$-classes</th>
<th>$\Gamma$-classes of $C_{2m}$</th>
<th>$[y]$</th>
<th>$[xy]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>CL_\alpha</td>
<td>$</td>
<td>1</td>
</tr>
<tr>
<td>$C_{Q_{128}}(CL_\alpha)$</td>
<td>$2^{h+2}$</td>
<td>$2^{h+2}$</td>
<td>$2^{h+1}$</td>
</tr>
</tbody>
</table>

| $\Phi_1$ | 0 | 0 |
| $\Phi_2$ | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $\Phi_f$ | 0 | 0 |
| $\Phi_{f+1}$ | $2^h$ | $2^h$ | 0 | 0 | $\ldots$ | 0 | 2 | 0 |
| $\Phi_{f+2}$ | $2^h$ | $2^h$ | 0 | 0 | $\ldots$ | 0 | 0 | 2 |

1.16 The Group $(Q_{2m} \times C_2)$ [10]
The direct product group $(Q_{2m} \times C_2)$ where $Q_{2m}$ is Quaternion group of order $4m$ with tow generators $x$ and $y$ is denoted by

$$Q_{2m} = \{x^k y^l : x^{2m} = y^4 = 1, y x^m y^{-1} = x^{-m}, 0 \leq k \leq 2m-1, l = 0,1\}$$

and $C_2$ is acyclic group of order 2 consisting of elements $\{1, z\}$ . the generalized the group $(Q_{2m} \times C_2)$ is denoted by

$$(Q_{2m} \times C_2) = \{(q,c) : q \in Q_{2m}, c \in C_2\}$$

and $|Q_{2m} \times C_2| = |Q_{2m}| \cdot |C_2| = 4m \cdot 2 = 8m$
2. The main results

In this section is to find the general form of Artin's characters table of the group $(Q_{2m} \times C_2)$ when $m=2^h, h \in \mathbb{Z}^+$

2.1 Proposition:
The general form of the Artin's characters table of the group $(Q_{2^h+1} \times C_2)$ when $m=2^h, h \in \mathbb{Z}^+$ is given as follows:

\[
\text{Ar}(Q_{2^h+1} \times C_2) = \begin{cases}
2\text{Ar}(Q_{2^{h+1}}) & \\
\text{Ar}(Q_{2^h+1}) & \\
\text{Ar}(Q_{2^{h+1}}) &
\end{cases}
\]

Table (7)

Proof:

Let $g \in (Q_{2^h+1} \times C_2); g=(q, I)$ or $g=(q, z), q \in Q^{2^h+1}, I, z \in C_2$

Case (I):
If $H$ is a cyclic subgroup of $(Q_{2^h+1} \times \{I\}), then:
1- $H=\langle (x, I) \rangle$
2- $H=\langle (y, I) \rangle$
3- $H=\langle (xy, I) \rangle$

And $\varphi$ the principal character of $H$, $\Phi_j$ Artin characters of $Q_{2^h+1}$ $1 \leq j \leq l+2$ then by using theorem (1.6)

1- $\Phi_j(g) = \frac{|C_g(g)|}{|C_H(g)|} \sum_{i=1}^{m} \varphi(h_i)$ if $h_i \in H \cap \text{CL}(g)$
2- $\Phi_j(g) = 0$ if $H \cap \text{CL}(g) = \phi$

If $g=(1, I)$

$$\Phi_{(1,I)}((1, I), g) = \frac{|C_{Q_{2^h+1} \times C_2}(g)|}{|C_H(g)|} \cdot \varphi(g) = 2^{h+3}$$

Since $H \cap \text{CL}(1, I) = \{(1, I)\}$

(ii) if $g=(x^{2^h}, I), g \in H$

$$\Phi_{(x^{2^h}, I), g}((x^{2^h}, I), g) = \frac{|C_{Q_{2^h+1} \times C_2}(g)|}{|C_H(g)|} \cdot \varphi(g) = 2^{h+3}$$
since $H \cap CL(g) = \{g, g^{-1}\}$ and $\varphi(g) = \varphi(g^{-1}) = 1$, $g = (q, I), q \in Q_2, q \neq x^{2i}$

(iv) if $g \notin H$

$\Phi_{(i,j)}(g) = 2.0 = 2.\Phi_{(i,j)}(q)$ Since $H \cap CL(g) = \emptyset$

2- IF $H = ((y, I)) = \{(1, I), (y, I), (y^2, I), (y^3, I)\}$

(i) If $g = (1, I)$

$H \cap CL(1, I) = \{(1, I)\}$

$\Phi_{(i+1, I)}(g) = \frac{C_{Q_2 \times C_2} (g)}{|C_{H}(g)|} \cdot 1 = \frac{8.2^h}{4} = \frac{2.2^h}{2} = 2.\Phi_{(i+1, I)}(g)$

(ii) If $g = (x^2, I) = (y^2, I)$ and $g \in H$

$\Phi_{(i+1, I)}(g) = \frac{C_{Q_2 \times C_2} (g)}{|C_{H}(g)|} \cdot \phi(g) = \frac{8.2^h}{4} = \frac{2.2^h}{2} = 2.\Phi_{(i+1, I)}(g)$

Since $H \cap CL(g) = \{g\}, \phi(g) = 1$

(iii) If $g \neq (x^2, I)$ and $g \in H$, i.e. $g = (y, I)$ or $g = (y^3, I)$

$\Phi_{(i+1, I)}(g) = \frac{C_{Q_2 \times C_2} (g)}{|C_{H}(g)|} \cdot (\phi(g) + \phi(g^{-1})) = \frac{8.2^h}{4} = \frac{2.2^h}{2} = 2.\Phi_{(i+1, I)}(g)$

since $H \cap CL(g) = \{g, g^{-1}\}$ and $\phi(g) = \phi(g^{-1}) = 1$

Otherwise

$\Phi_{(i+1, I)}(g) = 0$ since $H \cap CL(g) = \emptyset$

3- IF $H = ((xy, I)) = \{(1, I), (xy, I), ((xy)^2, I) = (y^2, I), ((xy)^3, I) = (y^3, I)\}$

(i) If $g = (1, I)$

$H \cap CL(1, I) = \{(1, I)\}$

$\Phi_{(i+1, I)}(g) = \frac{C_{Q_2 \times C_2} (g)}{|C_{H}(g)|} \cdot 1 = \frac{8.2^h}{4} = \frac{2.2^h}{2} = 2.\Phi_{(i+1, I)}(g)$

(ii) If $g = (x^2, I) = ((xy)^2, I) = (y^2, I)$ and $g \in H$

$\Phi_{(i+1, I)}(g) = \frac{C_{Q_2 \times C_2} (g)}{|C_{H}(g)|} \cdot \phi(g) = \frac{8.2^h}{4} = \frac{2.2^h}{2} = 2.\Phi_{(i+1, I)}(g)$

Since $H \cap CL(g) = \{g\}, \phi(g) = 1$

(iii) If $g \neq (x^2, I)$ and $g \in H$, i.e. $g = (xy, I)$ or $g = ((xy)^3, I)$

$\Phi_{(i+1, I)}(g) = \frac{C_{Q_2 \times C_2} (g)}{|C_{H}(g)|} \cdot (\phi(g) + \phi(g^{-1})) = \frac{8.2^h}{4} = \frac{2.2^h}{2} = 2.\Phi_{(i+1, I)}(g)$

since $H \cap CL(g) = \{g, g^{-1}\}$ and $\phi(g) = \phi(g^{-1}) = 1$

Otherwise

$\Phi_{(i+1, I)}(g) = 0$ since $H \cap CL(g) = \emptyset$

58
Case (II):
If H is a cyclic subgroup of \((Q_2^{h+1} \times \{z\})\), then:
1- \(H = \langle (x, z) \rangle\)
2- \(H = \langle (y, z) \rangle\)
3- \(H = \langle (xy, z) \rangle\)
And \(\varphi\) the principal character of \(H\), \(\Phi_j\) Artin characters of \(Q_2^{h+1}\) \(1 \leq j \leq l + 2\), then by using theorem (1.6)

1- \(\Phi_j(g) = \frac{C_{Q_{2^{h+1}}}^j(g)}{|C_H(g)|} \sum_{i=1}^{m} \varphi(h_i) \quad \text{if} \quad h_i \in H \cap CL(g)\)
2- \(\Phi_j(g) = 0 \quad \text{if} \quad H \cap CL(g) = \phi\)
1- \(\varphi\) the principal character of \(H\), \(\Phi_j\) Artin characters of \(Q_2^{h+1}\) \(1 \leq j \leq l\), then

(i) \(g=(1,1)\) or \(g=(1,z)\)

\[
\Phi_{(1,1)}(g) = \frac{|C_{Q_{2^{h+1}}}^j(g)|}{|C_H(g)|} \varphi(g) = \frac{2^{h+3}}{|C_H(g)|} \cdot \varphi(1) = \Phi_j(1) = \frac{2}{|C_H(g)|} \cdot \varphi(1) = \Phi_j(1) \quad \text{since} \quad H \cap CL(g) = \{1, (1,1), (1,z)\}
\]

(ii) if \(g=(1,1)\) or \(g=(x^{2^l}, I)\) or \(g=(x^{2^l}, z)\) or \(g=(1,z), g \in H\)
if \(g=(1,1)\) or \(g=(1,z)\)

\[
\Phi_{(1,1)}(g) = \frac{|C_{Q_{2^{h+1}}}^j(g)|}{|C_H(g)|} \varphi(g) = \frac{2^{h+3}}{|C_H(g)|} \cdot \varphi(g) = \frac{2}{|C_H(g)|} \cdot \varphi(g) = \Phi_j(g) \quad \text{since} \quad H \cap CL(g) = \{1, (1,1), (1,z)\}
\]

(iii) if \(g \neq (x^{2^l}, I)\) or \(g \neq (x^{2^l}, z)\), \(g \in H\)

\[
\Phi_{(1,1)}(g) = \frac{|C_{Q_{2^{h+1}}}^j(g)|}{|C_H(g)|} \varphi(g) + \varphi(g^{-1}) = \frac{2^{h+2}}{|C_H(g)|} (1 + 1) = \frac{2}{|C_H(g)|} \cdot \varphi(g) = \Phi_j(g) \quad \text{since} \quad H \cap CL(g) = \{1, g, g^{-1}\} \text{ and } \varphi(g) = \varphi(g^{-1}) = 1, g \in Q_{2^{h+1}} \text{ and } q \neq x^{2^l}
\]

(iv) if \(g \in H\)

\[
\Phi_{(1,1)}(g) = 0 \quad \text{Since} \quad H \cap CL(g) = \phi\)

2- IF \(H = \{(y, I), (y, y, I), (y^2, I), (y^3, I), (1, z), (y, z), (y^2, z), (y^3, z)\}\)
(i) \(g=(1,1)\) or \(g=(1,z)\)

\[
\Phi_{(1,1)}(g) = \frac{|C_{Q_{2^{h+1}}}^j(g)|}{|C_H(g)|} \varphi(g) = \frac{8.2^h}{8} \cdot 1 = 2^h = \Phi_{(1,1)}(1)
\]
If \( g = (x^{2^h}, I) = (y^2, I), (y^2, z) \) and \( g \in H \)
\[
\Phi_{(i+1,2)}(g) = \left[ \frac{C_{G_{x,y}, C_2}(g)}{C_H(g)} \right] \cdot \phi(g) = \frac{8.2^h}{8} \cdot 1 = 2^h = \Phi_{i+1}(x^{2^h})
\]
Since \( H \cap CL(g) = \{ g \} \), \( \phi(g) = 1 \)

(iii) If \( g \neq (x^{2^h}, I) \) and \( g \in H \), i.e. \( \{ g = (y, I), (y, z) \text{ or } g = (y^3, I), (y^3, z) \} \)
\[
\Phi_{(i+1,2)}(g) = \left[ \frac{C_{G_{x,y}, C_2}(g)}{C_H(g)} \right] \cdot (\phi(g) + \phi(g^{-1})) = \frac{8}{8} (1 + 1) = 2 = \Phi_{i+1}(y)
\]
since \( H \cap CL(g) = \{ g, g^{-1} \} \) and \( \phi(g) = \phi(g^{-1}) = 1 \)

Otherwise
\[
\Phi_{(i+1,2)}(g) = 0 \quad \text{since } H \cap CL(g) = \emptyset
\]

3. IF \( H = \langle (xy, I) = \{ (1, I), (xy, I), ((xy)^2, I) = (y^2, I), ((xy)^3, I) = (xy^3, I), (1, z), (xy, z), ((xy)^2, z), ((xy)^3, z) \rangle \)

(i) If \( g = (1, I) \) or \( g = (1, z) \)
\[
H \cap CL(g) = \{ g \}
\]
\[
\Phi_{(i+2,2)}(g) = \left[ \frac{C_{G_{x,y}, C_2}(g)}{C_H(g)} \right] \cdot \phi(g) = \frac{8.2^h}{8} \cdot 1 = 2^h = \Phi_{i+2}(1)
\]

(ii) If \( g = (x^{2^h}, I) = ((xy)^2, I) = (y^2, I), ((xy)^2, z) \) and \( g \in H \)
\[
\Phi_{(i+2,2)}(g) = \left[ \frac{C_{G_{x,y}, C_2}(g)}{C_H(g)} \right] \cdot \phi(g) = \frac{8.2^h}{8} \cdot 1 = 2^h = \Phi_{i+2}(x^{2^h})
\]
Since \( H \cap CL(g) = \{ g \} \), \( \phi(g) = 1 \)

(iii) If \( g \neq (x^{2^h}, I) \) and \( g \in H \), i.e. \( \{ g = (xy, I), ((xy)^3, I), (xy, z), ((xy)^3, z) \} \)
\[
\Phi_{(i+2,2)}(g) = \left[ \frac{C_{G_{x,y}, C_2}(g)}{C_H(g)} \right] \cdot (\phi(g) + \phi(g^{-1})) = \frac{8}{8} (1 + 1) = 2 = \Phi_{i+2}(xy)
\]
since \( H \cap CL(g) = \{ g, g^{-1} \} \) and \( \phi(g) = \phi(g^{-1}) = 1 \)

Otherwise
\[
\Phi_{(i+2,2)}(g) = 0 \quad \text{since } H \cap CL(g) = \emptyset
\]

2.2 Example:
To construct \( \text{Ar}(Q_{128} \times C_2) \) by using proposition (2.1) we have

1. \( \Phi_{(i,j)}(x^i, I) = 2 \Phi_j(x^i); x^i \in Q_{2m} \) when \( m = 2^h, h \in \mathbb{Z}^+ \), \( \Phi_{(i,j)}(y, I) = 2 \Phi_j(y) \), \( \Phi_{(i,j)}(x^i, I) = 2 \Phi_j(x^i) \) and \( \Phi_{(i,j)}(g) = 0 \) otherwise; \( g \in (Q_{2m} \times C_2) \) when \( m = 2^h, h \in \mathbb{Z}^+ \).
2. \( \Phi_{(i,j)}(x^i, I) = \Phi_j(x^i); x^i \in Q_{2m} \) when \( m = 2^h, h \in \mathbb{Z}^+ \), \( \Phi_{(i,j)}(y, I) = \Phi_j(y) \), \( \Phi_{(i,j)}(x^i, I) = \Phi_j(x^i) \) and \( \Phi_{(i,j)}(x^i, z) = \Phi_j(x^i); x^i \in Q_{2m} \) when \( m = 2^h, h \in \mathbb{Z}^+ \), \( \Phi_{(i,j)}(y, z) = \Phi_j(y) \), \( \Phi_{(i,j)}(x^i, y, z) = \Phi_j(x^i) \)
Then $\text{Ar}(Q_2 \times C_2) =$

| $\Gamma$-classes | $[\mathbb{I}]$ | $[x]^1 \mathbb{I}$ | $[x]^2 \mathbb{I}$ | $[x]^3 \mathbb{I}$ | $[x]^4 \mathbb{I}$ | $[x]^5 \mathbb{I}$ | $[x]^6 \mathbb{I}$ | $[y]$ | $[xy]$ | $[x]^2 \mathbb{I}$ | $[x]^3 \mathbb{I}$ | $[x]^4 \mathbb{I}$ | $[x]^5 \mathbb{I}$ | $[x]^6 \mathbb{I}$ | $[y]$ | $[xy]$ |
|------------------|---------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|-------|-------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|-------|-------|
| $[\mathbb{C} \mathbb{L}_a]$ | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 64 | 64 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 64 | 64 | 8 | 8 |
| $[\mathbb{C}_0 \ (\mathbb{C} \mathbb{L}_a)]$ | 512 | 512 | 256 | 256 | 256 | 256 | 256 | 8 | 8 | 512 | 512 | 256 | 256 | 256 | 256 | 256 | 256 | 256 | 256 | 256 | 8 | 8 |
| $\Phi_{(1,1)}$ | 512 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(2,1)}$ | 256 | 256 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(3,1)}$ | 128 | 128 | 128 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(4,1)}$ | 64 | 64 | 64 | 64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(5,1)}$ | 32 | 32 | 32 | 32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(6,1)}$ | 16 | 16 | 16 | 16 | 16 | 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(7,1)}$ | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(8,1)}$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(9,1)}$ | 128 | 128 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(10,1)}$ | 128 | 128 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(1,2)}$ | 256 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(2,2)}$ | 128 | 128 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(3,2)}$ | 64 | 64 | 64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(4,2)}$ | 32 | 32 | 32 | 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(5,2)}$ | 16 | 16 | 16 | 16 | 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(6,2)}$ | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(7,2)}$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(8,2)}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(9,2)}$ | 64 | 64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Phi_{(10,2)}$ | 64 | 64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
References