Further Results on Action of Finite Groups on Commutative Rings

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Abstract
Let R be a commutative ring with identity 1 and G be a finite group of automorphisms of R of order n. Let \( R^G \) be the fixed subring of R. In this paper we study the relations between the ideals of \( R^G \) and R and we study \( R^G \) in case R is a ring (field).

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Introduction
Let R be a commutative ring with identity 1 and let G be a finite group of automorphisms of R of order n. Let:
\[ R^G = \{ r \in R \mid g(r) = r, \text{ for all } g \in G \} \]
The set \( R^G \) is a subring of R, it is called the fixed ring of G. A.G.Naoum and the author [1,2] studied the relations between R and \( R^G \), they studied some certain ring theoretic properties of R which satisfied in \( R^G \), for more informations see [3,4].

In this paper we study some further results of the ring \( R^G \). We show that if I is G-invariant and \(|G|\) is invertible in R, then I is a maximal ideal in R if and only if \( I \cap R^G \) is a maximal ideal in \( R^G \), and if I is a prime ideal in R, then \( I \cap R^G \) is a prime ideal in \( R^G \).

Also, we show that if \((a,b)\) is a projective ideal of R, then \((a^m, b^n)\) is a projective ideal in \( R^G \), where \(|G| = n\).

Finally, we show that if R is e-ring (field) and \(|G|\) is invertible in R, then \( R^G \) is e-ring (field).

Ideals of R and \( R^G \):
In this section we study the relation between the elements, and the ideals of R and those of \( R^G \). We start with the following definition, remarks and proposition.

We recall that an ideal M of the ring R is said to be G-invariant if \( g(M) \subseteq M \), for all \( g \in G \), where \( g(M) = \{ g(m) \mid m \in M \} \).

Remarks:
1- If I is an ideals of R, then \( I \cap R^G \) is an ideal of \( R^G \).
2- If an element \( a \) in \( R^G \) is invertible in R, then \( a \) is invertible in \( R^G \).

The converse is clear [5].

Proposition:
Let G be a finite group of automorphisms of R of order n. Let \( b \in R \), and let
\[ x = \sum_{i=1}^{n} g_i(b), \quad y = \prod_{i=1}^{n} g_i(b) \quad \text{ and } \quad z \]

\[ = \sum_{i=1}^{n} g_i(b)g_{\delta(i)}(b), \text{ where } \delta \text{ is a 2-Cycle and } \]
\[ g_i \in G. \]

Then each of \( x \), \( y \) and \( z \) belong to \( R^G \), in general,
\[ \sum_{i=1}^{n} g_i(b)g_{\delta(i)}(b) \in R^G, \text{ where } \delta \text{ is an } m\text{-cycle and } 1 \leq m \leq n - 2. \]

Proof: [5].

Theorem:
Let R be a commutative ring with identity 1 and G be a finite group of automorphisms of R of order n and \(|G|\) is invertible in R. Let M be G-invariant ideal of R, then M is a maximal ideal in R if and only if \( M \cap R^G \) is a maximal ideal in \( R^G \).

Proof:
Let \( a \in R^G \) and \( a \notin M \cap R^G \), then \( a \notin M \). But M is maximal in R, so \( M + Ra = R \). Since \( l \in R \), then \( l \in M + Ra \), Thus \( l = m + ra \), where \( m \in M \), \( r \in R \). Thus:
\[ l = n^{-1}\left( \sum_{i=1}^{n} g_i(m) + a \sum_{i=1}^{n} g_i(r) \right) \]

But M is G-invariant, implies \( g_i(M) \subseteq M \), so \( g_i(m) \in M \), for all \( g_i \in G \) and then,
\[ n^{-1}\sum_{i=1}^{n} g_i(m) \in M. \]

By Proposition (1-2) and Remark (1-1), \( n^{-1}\sum_{i=1}^{n} g_i(m) \in R^G \),
implies \( \sum_{i=1}^{n} g_i(m) \in M \cap R^G \), and by Proposition (1-2), 
\( \sum_{i=1}^{n} g_i(r) \in R^G \), so \( \sum_{i=1}^{n} g_i(r) \in R^G a \).

Then \( 1 \in (M \cap R^G) + R^G a \). Therefore \( M \cap R^G \) is a maximal ideal in \( R^G \).

Conversely, let \( J \) be an ideal of \( R \) such that \( M \subseteq J \subseteq R \) then, \( M \cap R^G \subseteq J \cap R^G \subseteq R^G \). But \( M \cap R^G \) is a maximal ideal in \( R^G \), so \( J \cap R^G = R^G \). Since \( 1 \in R^G \), then \( 1 \in J \cap R^G \) which means \( 1 \in J \). Hence \( J = R \). Therefore \( M \) is a maximal ideal in \( R \).

**Theorem:**

Let \( R \) be a commutative ring with identity 1 ring and \( G \) be a finite group of automorphisms of \( R \) of order \( n \). If \( I \) is a prime ideal in \( R \), then \( I \cap R^G \) is a prime ideal in \( R^G \).

**Proof:**

Let \( x, y \in R^G \) such that \( x, y \in I \cap R^G \), then \( x, y \in I \) and \( x, y \in R^G \). Since \( I \) is a prime ideal in \( R \), then either \( x \in I \) or \( y \in I \). Hence either \( x \in I \cap R^G \) or \( y \in I \cap R^G \); then \( I \cap R^G \) is a prime ideal in \( R^G \).

Before we start the next result, we recall that a finitely generated ideal \( A \) which is generated by \( \{a_1, a_2, ..., a_n\} \) in \( R \) is projective if and only if there exists an \( n \times n \) matrix \( M = (r_{ij}) \) with elements in \( R \) such that:

i) \( Um = u \), and

ii) \( U^T = \text{ann}(M) \)

where \( U = (a_1, a_2, ..., a_n) \in R^n \) is a vector and \( U^T = \{X \in R^n ; UX = 0\} \) is the orthogonal complement of \( U \). \( X^T \) is the column vector which is the transpose of \( X \).[6]

**Theorem:**

If the ideal \( (a, b) \) is a projective in \( R \), then \( (a^b, b^a) \) is a projective in \( R^G \).

**Proof:**

Let \( (a, b) \) be an ideal in \( R^G \), implies that \( a, b \in R \) and \( (a, b) \) is a projective in \( R \),

then there exists a matrix \( M = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \)

where \( r_{ij} \in R, i, j = 1, 2 \) such that:

1) \( (a, b) M = (a, b) \) and

2) \( \text{ann}(a, b) = \text{ann}(M) \), thus:

\( (a, b), \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = (a, b) \)

Hence \( ar_{11} + br_{12} = a \)

\( ar_{21} + br_{22} = b \). Thus:

\( a(1-r_{11}) = b(1-r_{22}) = ar_{12} \). Therefore:

\[ a^n \prod_{i=1}^{n} \left( 1 - g_i(r_{11}) \right) = b^n \prod_{i=1}^{n} \left( 1 - g_i(r_{22}) \right) = (a, b) \]

Put \( \prod_{i=1}^{n} \left( 1 - g_i(r_{11}) \right) = 1 - s_{11} \prod_{i=1}^{n} g_i(r_{21}) = s_{11} \)

\[ \prod_{i=1}^{n} \left( 1 - g_i(r_{22}) \right) = 1 - s_{22} \prod_{i=1}^{n} g_i(r_{12}) = s_{22} \]

Then \( 1-s_{11}, s_{11} = 1 - s_{22}, \) and \( s_{12} \) are in \( R^G \) (by Proposition 1-2).

Thus \( a^n - a^n s_{11} + b^n s_{21} \) and \( b^n - b^n s_{22} + a^n s_{12} \)

Put \( M' = \begin{pmatrix} x \\ y \end{pmatrix} . \) Hence \((a^n, b^n) M' = (a^n, b^n) \).

Now to prove that \( \text{ann}(a^n, b^n) \subseteq \text{ann}(M') \).

Let \( (x, y) \in \text{ann}(a^n, b^n) \).

To prove \((x, y) \in \text{ann}(M') \), use induction on \( |G| \): If \( n = 1 \), \( (x, y) \in \text{ann}(a, b) \) implies \((x, y) \in \text{ann}(M) \). So:

\[ M' = \begin{pmatrix} x \\ y \end{pmatrix} \]

Hence \( r_{11} x + r_{22} y = 0 \) and \( r_{12} x + r_{22} y = 0 \)

But \( |G| = 1 \), so \( R = R^G \) and \( M = M' \). Thus \((x, y) \in \text{ann}(M') \).

Suppose it is true for \( n-1 \) that is \( \text{ann}(a^{n-1}, b^{n-1}) \subseteq \text{ann}(M) \).

Let \((x, y) \in \text{ann}(a^n, b^n) \), then \((xa^{n-1}, yb^{n-1}) \in \text{ann}(a, b) \). So \((xa^{n-1}, yb^{n-1}) \in \text{ann}(M) \).

Thus \( s_{11} xa^{n-1} + s_{21} yb^{n-1} = 0 \), \( s_{12} xa^{n-1} + s_{22} yb^{n-1} = 0 \).

Therefore \( (s_{11}, s_{22}) \in \text{ann}(a^n, b^n) \subseteq \text{ann}(M) \) and \( (s_{12}, s_{22}) \in \text{ann}(a^n, b^n) \subseteq \text{ann}(M) \).

Hence:

\( s_{11} (s_{11} x) + s_{21} (s_{22} y) = 0 \)

\( s_{12} (s_{21} x) + s_{22} (s_{22} y) = 0 \).

Thus, \( s_{11} = (1 - s_{11}) \prod_{k=1}^{n-1} g_i(r_{11}) \)

\[ = \prod_{k=1}^{n-1} g_i(r_{11}) \cdot g_i(r_{11}) \]

Hence

\[ g(r_{11}) \prod_{k=1}^{n-1} g_i(r_{11}) = \prod_{k=1}^{n-1} g_i(r_{11}) = s_{11} \]

Similarly for \( s_{12} \) \( s_{21} \) and \( s_{22} \): Hence:

\( s_{11} x + s_{22} y = 0 \), and \( s_{12} x + s_{22} y = 0 \)

Thus \((x, y) \in \text{ann}(M) \) and \( \text{ann}(M') \subseteq \text{ann}(a^n, b^n) \).

Therefore \( \text{ann}(a^n, b^n) = \text{ann}(M) \). So \((a^n, b^n) \) is a projective ideal in \( R^G \).
e-ring and fields:
We start this section by the following:
We recall that a ring $R$ is said to be e-ring if for all $x \in R$, there exists $y \in R$ such that $xy = x$ [7].

**Theorem:**
Let $R$ be a commutative ring with identity 1 and $G$ be a finite group of automorphisms of $R$ of order $n$, and $|G|$ is invertible in $R$. If $R$ is an e-ring, then $R^G$ is an e-ring.

**Proof:**
Let $x \in R^G$, then $x \in R$, but $R$ is an e-ring, then there exist $y \in R$, such that $xy = x$. Thus $x n \sum_{i=1}^{n} g_i(y) = nx$, $g_i \in G$. So $x(n^{-1} \sum_{i=1}^{n} g_i) = x$. This means $n^{-1} \sum_{i=1}^{n} g_i(y) \in R^G$ [by Proposition 1-2 and Remark 1-l(2)]. Therefore $R^G$ is an e-ring. Finally, we have the following result.

**Theorem:**
Let $R$ be a commutative ring with identity 1 and $G$ be a finite group of automorphisms of $R$ of order $n$, such that $|G|$ is invertible in $R$. If $R$ is a field, then $R^G$ is a field.

**Proof:**
$R$ is a commutative ring with 1, implies that $R^G$ is a commutative ring with 1. Let $r \in R^G$, $r \neq 0$ then $r \in R$. Hence, there exists $s \in R$ such that $rs = 1$, then $r \sum_{k=1}^{n} g_k(s) = n$, Since $|G|$ is invertible in $R$, then $r(n^{-1} \sum_{i=1}^{n} g_i(s)) = 1$. By Remark 1-1(2) and Proposition 1-2, $n^{-1} \sum_{i=1}^{n} g_i(s) = r^{-1} \in R^G$, then $r$ has a multiplicative inverse in $R^G$. Therefore $R^G$ is a field.

**References:**