Some Convergence Theorems for the Fixed Point in Banach Spaces

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Abstract

Let X be a uniformly smooth Banach space, T:X → X be Φ-strongly quasi accretive (Φ-hemi contractive) mappings. It is shown under suitable conditions that the Ishikawa iteration sequence converges strongly to the unique solution of the equation Tx = f. Our main results is to improve and extend some results about Ishikawa iteration for type from contractive, announced by many others.

Keywords: Convergence Theorems, Fixed Point, Banach Spaces

1. Introduction

Let X be an arbitrary Banach space with norm ||·|| and the dual space X*. The normalized duality mapping J:X → 2 is defined by

J(x) = {f ∈ X*: < x, f > := || x || || f ||, || f || = || x ||}

where <·,·> denotes the normalized duality pairing. It is known that if X is uniformly smooth, then J is single valued and is uniformly continuous on any bounded subset of X.

Let T:D(T) ⊆ X → X be an operator, where D(T) and R(T) denote the domain and range of T, respectively, and I denote the identity mapping on X.

We recall the following two iterative processes to Ishikawa and Mann, [1], [2]:

i- Let K be a nonempty convex subset of X, and T:K → K be a mapping, for any given x0 ∈ K the sequence <xn> defined by

xn + 1 = (1 - αn)xn + αnTyn + un (n ≥ 0)

is called Ishikawa iteration sequence, where <αn> and <βn> are two real sequences in [0,1] satisfying some conditions.

ii- In particular, if βn = 0 for all n ≥ 0 in (i), then <xn> defined by

x0 ∈ K, xn + 1 = (1 - αn)xn + αnTyn, n ≥ 0

is called the Mann iteration sequence.

Recently Liu [3] introduced the following iteration method which he called Ishikawa (Mann) iteration method with errors.

For a nonempty subset K of X and a mapping T:K → K, the sequence <xn> defined for arbitrary x0 ∈ K by

yn = (1 - βn)xn + βnTyn + vn,

xn + 1 = (1 - αn)xn + αnTyn + un for all n = 0, 1,2,...

where <un> and <vn> are two summable sequences in X (i.e., ||un|| < ∞ and ||vn|| < ∞), <αn> and <βn> are two real sequences in [0,1], satisfying suitable conditions, is called the Ishikawa iterates with errors. If βn = 0 and vn = 0 for all n, then the sequence <xn> is called the Mann iterates with errors.

The purpose of this paper is to define the Ishikawa iterates with errors to fixed points and solutions of Φ-strongly quasi accretive and Φ-hemi-contractive operators equations. Our main results improve and extend the corresponding results recently obtained by [4] and [5]. Via replaced the assumption summable sequences by the assumption bounded sequences, T need not be Lipschitz and the assumption that T is strongly accretive mapping is replaced by assumption that T is Φ-strongly quasi accretive and Φ-hemi-contractive, and main results improve the corresponding results recently obtained by [6]. Via replaced the assumption quasi-strongly accretive and quasi-strongly pseudo-contractive mappings by the assumption Φ-strongly quasi accretive and Φ-hemi-contractive operators.

1.1 Definition: [6], [7]

A mapping T with domain D(T) and range R(T) in X is said to be strongly accretive if for any x, y ∈ D(T), there exists a constant k ∈ (0,1) and j(x - y) ∈ J(x - y) such that <Tx - Ty, j(x - y)> ≥ k ||x - y||^2.

The mapping T is called Φ-strongly accretive if there exists a strictly increasing function Φ : [0,∞) → [0,∞] with Φ(0) = 0 such that the inequality

<Tx - Ty, j(x - y)> ≥ Φ(||x - y||)||x - y||

holds for all x, y ∈ D(T). It is well known that the class of strongly accretive mappings is a proper
subclass of the class of \( \Phi \)-strongly accretive mapping.

An operator \( T: X \rightarrow X \) is quasi-strongly accretive if there exists a strictly increasing function \( \Phi: [0, \infty) \rightarrow [0, \infty) \) with \( \Phi(0) = 0 \) such that for any \( x, y \in D(T) \)
\[
\text{Re} \langle Tx - Ty, j(x - y) \rangle \geq \Phi(\|x - y\|)
\]

An operator \( T: X \rightarrow X \) is called \( \Phi \)-strongly quasi-accretive if there exist a strictly increasing function \( \Phi: [0, \infty) \rightarrow [0, \infty) \) with \( \Phi(0) = 0 \) such that for all \( x \in D(T) \), \( p \in N(T) \) there exist \( j(x - p) \in J(x - p) \) such that
\[
\langle Tx - Ty, j(x - y) \rangle \geq \Phi(\|x - y\|) \cdot \|x - p\|.
\]

1.2 Remarks: [5], [6], [7]
1. A mapping \( T: X \rightarrow X \) is called strongly pseudo contractive if and only if \( (I - T) \) is strongly accretive.
2. A mapping \( T: X \rightarrow X \) is called \( \Phi \)-strongly pseudo-contractive if and only if \( (I - T) \) is \( \Phi \)-strongly accretive.
3. A mapping \( T: X \rightarrow X \) is called quasi-strongly pseudo-contractive if and only if \( (I - T) \) is quasi-strongly accretive.
4. A mapping \( T: X \rightarrow X \) is called \( \Phi \)-hemicontractive if and only if \( (I - T) \) is \( \Phi \)-strongly quasi-accretive.

The following lemma plays an important role in proving our main results.

1.1 Lemma: [1], [2]
Let \( X \) be a Banach space. Then for all \( x, y \in X \) and \( j(x + y) \in J(x + y) \),
\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.
\]

2. Main Results

Now, we state and prove the following theorems:

2.1 Theorem:
Let \( X \) be a uniformly smooth Banach space and let \( T: X \rightarrow X \) be a \( \Phi \)-strongly quasi-accretive operator.
Let \( x_0 \in K \) the Ishikawa iteration sequence \( \langle x_n \rangle \) with errors be defined by
\[
y_n = (1 - \beta_n)x_n + \beta_nSx_n + b_nv_n
\]
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nSy_n + a_nu_n \quad \text{for all } n = 0, 1, 2, \ldots
\]
where \( \langle \alpha_n \rangle, \langle \beta_n \rangle, \langle a_n \rangle \) and \( \langle b_n \rangle \) are sequences in \([0,1]\) satisfying
\[
\lim_{n \to \infty} \alpha_n = 0 \quad \text{and} \quad \lim_{n \to \infty} \beta_n = 0
\]
\[
\sum_{n=0}^{x} \alpha_n = \infty \quad \text{and} \quad a_n \leq \alpha_n^{1+c}, \quad c > 0, \quad b_n > 1
\]
and \( \langle u_n \rangle \) and \( \langle v_n \rangle \) are two bounded sequence in \( X \).
Define \( S: X \rightarrow X \) by
\[
Sx = f + x - Tx \quad \text{for all } x \in X, \quad \text{and suppose that } R(S) \text{ is bounded, then}
\]
\( \langle x_n \rangle \) converges strongly to the unique solution of the equation \( Tx = f \).

**proof:** Since \( T \) is \( \Phi \)-strongly quasi-accretive, it follows that \( N(T) \) is a singleton, say \( \{ w \} \).
Let \( T w = f \), it is easy to see that \( S \) has a unique fixed point \( w \), it follows from definition of \( S \) that
\[
\langle Sx - Sy \rangle = \langle x - y \rangle
\]
and hence
\[
\langle x - y \rangle \leq \langle x - w \rangle - \Phi(\|x - y\|) \cdot \|x - w\| \ldots (6)
\]
Setting \( y = w \), we have
\[
\langle Sx - Sw \rangle \leq \langle x - w \rangle - \Phi(\|x - w\|) \cdot \|x - w\| \ldots (7)
\]
We prove that \( \langle x_n \rangle \) and \( \langle y_n \rangle \) are bounded. Let
\[
M_1 = \sup \{ \|Sx_n - w\| + \|Sy_n - w\| : n \geq 0 \} + \|x_0 - w\|
\]
\[
M_2 = \sup \{ \|u_n\| + \|v_n\| : n \geq 0 \}
\]
From (2) and (5), we get
\[
\|x_{n+1} - w\| \leq (1 - \alpha_n)\|x_n - w\| + \alpha_n\|Sy_n - w\| + a_n\|u_n\| \leq (1 - \alpha_n)\|x_n - w\| + \alpha_nM_1 + \alpha_nM_2
\]
and hence
\[
\|x_{n+1} - w\| \leq (1 - \alpha_n)\|x_n - w\| + \alpha_nM \quad \ldots (8)
\]
Now, from (1) and (5), we have
\[
\|y_n - w\| \leq (1 - \beta_n)\|x_n - w\| + \beta_n\|Sx_n - w\| + b_n\|v_n\| \leq (1 - \beta_n)\|x_n - w\| + \beta_nM_1 + \beta_nM_2
\]
and hence
\[
\|y_n - w\| \leq (1 - \beta_n)\|x_n - w\| + \beta_nM \quad \ldots (9)
\]
\[
\|x_n - w\| \leq M \quad \ldots (10)
\]
Now, we show by induction that for all \( n \geq 0 \). For \( n = 0 \) we have
\[
\|x_0 - w\| \leq M_1 \leq M, \quad \text{by definition of } M_1 \text{ and } M.
\]
Assume now that \( \|x_n - w\| \leq M \) for some \( n \geq 0 \). Then by (8), we have
\[
\|x_{n+1} - w\| \leq (1 - \alpha_n)\|x_n - w\| + \alpha_nM \leq (1 - \alpha_n)M + \alpha_nM = M.
\]
Therefore, by induction we conclude that (10) holds substituting (10) into (9), we get
\[
\|y_n - w\| \leq M
\]
From (9), we have
\[
\|y_n - w\|^2 \leq (1 - \beta_n)^2\|x_n - w\|^2 + 2\beta_n(1 - \beta_n)M\|x_n - w\|^2 + \beta_n^2M^2
\]
Since \( 1 - \beta_n \leq 1 \) and \( \|x_n - w\| \leq M \), we get
\[
\|y_n - w\|^2 \leq \|x_n - w\|^2 + 2\beta_nM^2 \quad \ldots (12)
\]
Using lemma (1.1), we get
\[ \|x_{n+1} - w\|^2 \leq \|x_n - w\|^2 - \alpha_n \|Sy_n - w\|^2 + \beta_n \|y_n - w\|^2 + \lambda_n \|y_n - w\| \]

\[ \leq \|x_n - w\|^2 - \alpha_n \|Sy_n - w\|^2 + \alpha_n \|y_n - w\|^2 + \beta_n \|y_n - w\|^2 + 2\alpha_n \|y_n - w\| + 2\beta_n \|y_n - w\| \]

where 
\[ c_n = \langle Sy_n - w, j(x_{n+1} - w) - j(y_n - w) \rangle \]

By (10) and (12) and using that \( a_n \leq \alpha_n \beta_n \), we obtain 
\[ \|x_{n+1} - w\|^2 \leq \|x_n - w\|^2 - 2\alpha_n \|x_n - w\|^2 + \alpha_n^2 M^2 + 2\beta_n \|x_n - w\|^2 + 2\alpha_n \Phi(\|y_n - w\|) \]

and hence 
\[ \|x_{n+1} - w\|^2 \leq \|x_n - w\|^2 - 2\alpha_n \Phi(\|y_n - w\|) + \|y_n - w\| + \alpha_n \lambda_n \]

where \( \lambda_n = (\alpha_n + 2\alpha_n^2 + 4\beta_n)M^2 + 2c_n \).

First we show that \( c_n \to 0 \) as \( n \to \infty \), observe that from (1) and (2), we have 
\[ \|x_{n+1} - y_n\| \leq \|x_n - y_n\| + \|y_n - w\| \leq \|x_n - y_n\| + \|y_n - w\| \]

and 
\[ \|x_{n+1} - y_n\| \leq \|x_n - y_n\| + \|y_n - w\| \]

hence, by (10) and definition of \( M \), 
\[ \|x_{n+1} - y_n\| \leq (3\beta_n + \alpha_n)M \]

Therefore 
\[ \|x_{n+1} - w - (y_n - w)\| \to 0 \] as \( n \to \infty \).

Since \( \langle x_{n+1} - w, y_n - w \rangle \) and \( \langle Sy_n - w \rangle \) are bounded, and \( j \) is uniformly continuous on any bounded subset of \( X \), we have 
\[ j(x_{n+1} - w) - j(y_n - w) \to 0 \] as \( n \to \infty \),
\[ c_n = \langle Sy_n - w, j(x_{n+1} - w) - j(y_n - w) \rangle \to 0 \]

as \( n \to \infty \). Thus \( \lim_{n \to \infty} \lambda_n = 0 \).

inf \[ \|y_n - w\| : n \geq 0 \] = \( S \geq 0 \).

We prove that \( S = 0 \). Assume the contrary, i.e., \( S > 0 \).

Then 
\[ \|y_n - w\| \geq S > 0 \]

for all \( n \geq 0 \).

Hence 
\[ \Phi(\|y_n - w\|) \geq \Phi(S) > 0 \]

Thus from (14)
\[ \|x_{n+1} - w\|^2 \leq \|x_n - w\|^2 - \alpha_n \Phi(S) \cdot S - \|y_n - w\|^2 \]

for all \( n \geq 0 \). Since \( \lim_{n \to \infty} \lambda_n = 0 \), there exists a positive integer \( n_0 \) such that \( \|x_n - w\|^2 \leq \|x_n - w\|^2 - \alpha_n \Phi(S) \cdot S - \|y_n - w\|^2 \)

Therefore, from (16), we have 
\[ \|x_{n+1} - w\|^2 \leq \|x_n - w\|^2 - \alpha_n \Phi(S) \cdot S \]

or 
\[ \alpha_n \Phi(S) \cdot S \leq \|x_n - w\|^2 - \|x_{n+1} - w\|^2 \] for all \( n \geq n_0 \).

Hence 
\[ \Phi(S) \cdot S \leq \sum_{n=n_0}^{\infty} \alpha_n \|x_n - w\|^2 - \|x_{n+1} - w\|^2 \] 

which implies \( \sum_{n=0}^{\infty} \alpha_n < \infty \), contradicting (4).

Therefore, \( S = 0 \).

From definition of \( S \), there exists a subsequence \( <\|y_n - w\| > \), which we will denote by
\[ <\|y_i - w\| > \] such that
\[ \lim_{j \to \infty} \|y_j - w\| = 0 \]

Observe that from (1) for all \( n \geq 0 \), we have 
\[ \|x_n - w\| \leq \|y_n - w + \beta_n (x_n - w) - \beta_n (Sx_n - w) + b_n \| \]

\[ \leq \|y_n - w\| + \beta_n \|x_n - w\| + b_n \|Sx_n - w\| \]

Since \( b_n \leq \beta_n \), by definition of \( A, B \) and \( M \) we get
\[ \|x_n - w\| \leq \|y_n - w\| + 3\beta_n M \]

for all \( n \geq 0 \) ... (18)

Thus by (3), (17) and (18), we have
\[ \lim_{j \to \infty} \|x_j - w\| = 0 \]

Let \( \epsilon > 0 \) be arbitrary. Since \( \lim_{n \to \infty} \alpha_n = 0, \lim_{n \to \infty} \beta_n = 0 \)

and \( \lim_{n \to \infty} \lambda_n = 0 \), there exists a positive integer \( N_0 \) such that
\[ \alpha_n \leq \frac{\epsilon}{3M}, \beta_n \leq \frac{\epsilon}{3M}, \lambda_n \leq \frac{\epsilon}{3} \] for all \( n \geq N_0 \).

From (19), there exists \( k \geq N_0 \) such that
\[ \|x_k - w\| < \epsilon \]

We prove by induction that
\[ \|x_{k+n} - w\| < \epsilon \] for all \( n \geq 0 \)

For \( n = 0 \) we see that (21) holds by (20).

Suppose that (21) holds for some \( n \geq 0 \) and that
\[ \|x_{k+n+1} - w\| \geq \epsilon \]. Then by (15), we get
\[ \varepsilon \leq \| y_{k,n} - w \| - \| y_{k,n} - w + x_{k,n} - y_{k,n} \| \]
\[ \leq \| y_{k,n} - w \| + \| x_{k,n} - Y_{k,n} \| \]
Hence
\[ \leq \| y_{k,n} - w \| + (\alpha_{k,n} + 3\beta_{k,n}) M \]
\[ \leq \| y_{k,n} - w \| + \frac{2\varepsilon}{3} \]
\[ \| y_{k,n} - w \| \geq \frac{\varepsilon}{3} \]
From (14), we get
\[ \varepsilon \leq \| y_{k,n} - w \| - \| y_{k,n} - w - 2\alpha_{k,n} \phi(\varepsilon) \| \]
\[ \leq \alpha_{k,n} \phi\left( \frac{\varepsilon}{3} \right) \frac{\varepsilon}{3} \]
\[ = \| y_{k,n} - w \| - \varepsilon^2 < \varepsilon^2 \]
which is a contradiction. Thus we proved (21). Since \( \varepsilon \) is arbitrary, from (21), we have
\[ \lim_{n \to \infty} \| x_n - w \| = 0. \]

2.1 Remark:
If in theorem (2.1), \( \beta_n = 0, b_n = 0 \), then we obtain a result that deals with the Mann iterative process with errors.

Now, we state the Ishikawa and Mann iterative process with errors for the \( \Phi \)-hemicontractive operators.

2.2 Theorem:
Let \( X \) be a uniformly smooth Banach space, let \( K \) be a non empty bounded closed convex subset of \( X \) and \( T: K \to K \) be a \( \Phi \)-hemicontractive operator. Let \( w \) be a fixed point of \( T \) and let for \( x_0 \in K \) the Ishikawa iteration sequence \( \{ x_n \} \) be defined by
\[ y_n = \beta_n x_n + \beta_n T x_n + b_n v_n \]
\[ x_{n+1} = \alpha_n x_n + \alpha_n T y_n + a_n u_{n+1} , \quad n \geq 0 \]
where \( \{ u_n \}, \{ v_n \} \subseteq K, \{ \alpha_n \}, \{ \beta_n \}, \{ a_n \}, \{ b_n \} \) are sequences as in theorem (2.1) and
\[ \alpha_n = 1 - \alpha_n - a_n, \]
\[ \beta_n = 1 - \beta_n - b_n. \]
Then \( \{ x_n \} \) converges strongly to the unique fixed point of \( T \).

**proof:** Obviously \( \{ x_n \} \) and \( \{ y_n \} \) are both contained in \( K \) and therefore, bounded. Since \( T \) is \( \Phi \)-hemicontractive, then \( (I - T) \) is \( \Phi \)-strongly quasi accretive. The rest of the proof is identical the proof of theorem 2.1 with \( y = w \) and \( T = S \), and is therefore omitted. ■

2.2 Remark:
If in theorem (2.2), \( \beta_n = 0 \) and \( b_n = 0 \), then we obtain the corresponding result for the Mann iteration process with errors.

**References**