A Note on Pure Submodule Relative to Submodule

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Abstract
In this paper we used the concept of a pure submodule relative to submodule T in two concepts, pure relative to submodule T Baer injective modules and module with pure relative to submodule T intersection property. Some properties and some characterization of this notions are established.

Keywords: pure submodule, T-pure submodule, T-pure Baer injective module.

Introduction
Let R be associative ring with a non-zero identity and R-module will mean unitary left R-module. Recall that a submodule N of an R-module M is pure submodule if for every finitely generated ideal of R, N∩ IM = IN [1]. Following [2] a submodule N of an R-module M is pure submodule relative to submodule T of M (simply T-pure) if N∩IM=IN+T∩(N∩IM) for every ideal I in R. Every pure submodule is T-pure submodule but the converse is not true for example see [2]. An R-module M is called a pure Baer injective module, if for each pure left ideal A of R, any R-homomorphism f : A → M can be extended to an R-homomorphism h: R → M [3].

In this paper we introduce the concept of pure relative to submodule Baer injective modules (simply T-pure Baer injective). In [4] modules with the intersection property of any two pure submodule is pure (simply PIP). This led us to introduce the concept of a module with the property that the intersection of any two T-pure submodules is T-pure submodule.

1- Pure Relative ToSubmodule Baer Injective Modules.
Now we introduce the concept of pure relative to submodule Baer injective modules (simply T-pure Baer injective).

Definition 1.1 : [2]
Let M be an R-module and T be a submodule of M. A submodule N of M is said to be T-pure if for each ideal I of R, N∩IM = IN +T∩(N∩IM).

Let T be an ideal in R, a left ideal A of R is said to be T-pure if for every x ∈A there exists y ∈ T ∩ A.

Now we give some properties of T-pure submodules.

Remark 1.2 :
1. Let M be an R-module and let N be T-pure submodule of M. If H is T-pure submodule of N, then H is T-pure submodule of M.
2. Let M be an R-module and let N be T-pure submodule of M. If A is a submodule of M containing N, then N is a T-pure submodule of A.
3. Let M be an R-module and let N be T-pure submodule of M. If H is a submodule of N and H is submodule of T, then N

Proof:
1- Let I be an ideal of R, since N is T-pure in M and H is T-pure in N, then N[ boyfriend ] IM = IN + T[ boyfriend ] (N[ boyfriend ] IM) and hence H[ boyfriend ] IM ⊆ [IN+T[ boyfriend ] (N[ boyfriend ] IM)].

2- Let I be an ideal of R, since N is T-pure in M, then N[ boyfriend ] IM = IN+T[ boyfriend ] (N[ boyfriend ] IM).
Definition 1.3:
Let M be an R-module and T be a submodule of M. M is called T-pure Baer injective module if for each T-pure ideal A of R, any R-homomorphism f : A → M, there exists R-homomorphism h : R → M such that f(a) = h(a) for all a ∈ A.

Clearly, an R-module is pure Baer injective if and only if M is (0)-pure Baer injective. If M is a T1-pure Baer injective R-module, then M is T2-pure Baer injective for each submodule T2 containing T1. Thus every pure Baer injective module is T-pure Baer injective R-module.

Now we give another caretrization of T-pure Baer injective modules.

Theorem 1.4:
For an R-module M the following are equivalent:
1- M is T-pure Baer injective,
2- For T-pure left ideal A of R and every R-homomorphism f : A → M there exists m ∈ M such that for all a ∈ A, am = f(a)

Proof: Clear

Proposition 1.5:
If the direct product \( \prod_{\alpha} M_\alpha \) of R-modules is \( J_\alpha(T) \)-pure Baer injective, where \( J_\alpha \) is pinjection of \( M_\alpha \) into \( \prod_{\alpha} M_\alpha \) then \( M_\alpha \) is T-pure Baer injective for each \( \alpha \).

Proof:
Let A be a T-pure submodule of \( M_\alpha \) and f : A → \( M_\alpha \) be R-homomorphism. Since \( \prod_{\alpha} M_\alpha \) is \( J_\alpha(T) \)-pure Baer injective, there exists \( \rho_\alpha \) such that for all \( \alpha \), \( \rho_\alpha \circ f(\alpha) \in J_\alpha(T) \cap J_\alpha \circ f(A) \). Since \( \rho_\alpha \circ f(\alpha) \in J_\alpha(T) \cap J_\alpha \circ f(A) \), thus \( \rho_\alpha \circ f(\alpha) \in J_\alpha \circ f(A) \). Hence \( M_\alpha \) is T-pure Baer injective module for each \( \alpha \).

Recall that an R-module P is projective, if given any R-epimorphism \( f : A → B \), there exists R-homomorphism \( g : M → B \) can be lifted to an R-homomorphism \( h : M → A \). [5]

Theorem 1.6:
If every T-pure ideal of R is projective. Then the homomorphic image of a T-pure Baer injective module is t-pure Baer injective.

Proof:
Consider the following diagram of R-modules:

\[
\begin{array}{ccc}
0 & \rightarrow & M \\
\downarrow \ g & & \downarrow \ h \\
T & \rightarrow & K \\
\downarrow \ h_1 & & \downarrow \ f \\
0 & \rightarrow & A \\
\end{array}
\]

Where A is left T-pure ideal of R and i is the inclusion map and M is T-pure Baer injective module. Projectivity of A shows that for some R-homomorphism \( h : A → M \) ther is R-homomorphism \( h: A → M \) such that \( i(a) = f(a) \) for all \( a ∈ A \).

Since M is T-pure Baer injective module, there exists \( h_1 : R → M \) such that \( i(\alpha) \in T \cap f(A) \). Hence \( h_1 \circ i(a) ∈ T \cap f(A) \). Therefore K is T-pure Baer injective.

The converse of the above theorem is not true in general. We need the following concept, let M be an R-module and T a submodule of M, M is said to be projective relative to submodule T (simply T-projective), if for each R-epimorphism \( f : A → B \), there exists R-homomorphism \( g : M → B \) such that \( g(\alpha) \in T \cap f(A) \).
Let $A_1, B_1$ be two $\frac{T}{N}$ pure submodules of $\frac{M}{N}$ and let $K$ be an ideal in $R$. We want to show that 
$$\left( \frac{A_1}{N} \right) \cap \left( \frac{B_1}{N} \right) = K \left( \frac{M}{N} \right),$$
and this implies that 
$$\frac{A_1(NIM+N)}{N} = \frac{IA_1+N}{N} + \frac{(T+N)(An(N+M+N))}{N},$$
therefore, 
$$A \left( \frac{I(NIM+N)}{N} \right) = IA_1+T \left( \frac{IA_1}{N} \right) + N,$$
and hence $(A \left( \frac{I}{N} \right) + N = IA_1 + T \left( \frac{IA_1}{N} \right) + N).$ Since 
$x \in A \left( \frac{I}{N} \right) \subseteq A \left( \frac{I}{N} \right) (IM+N)$, then 
$x \in IA + T \left( \frac{IA_1}{N} \right) + N$ 
Let $x = w + m + n$, where $w \in IA$ and $m \in T \left( \frac{IA_1}{N} \right)$ and $n \in N$.
Now, consider $n = x - w - m \in N \left( \frac{I}{N} \right) IM = IN + T \left( \frac{I}{N} \right) N(IM) \subseteq IA + T \left( \frac{IA_1}{N} \right)$ 
And hence $A$ is $T$-pure in $M$. Since $M$ has the $T$-PIP, then $A \left( \frac{B}{N} \right)$ is $T$-pure in $M$.
Thus $(A \left( \frac{B}{N} \right) \cap KM = K \left( A \left( \frac{B}{N} \right) + T \left( \frac{IA_1}{N} \right) \right)$. 
Now, let $x \in \left( \frac{A_1}{N} \right) \cap \left( \frac{B_1}{N} \right) \cap K \left( \frac{M}{N} \right)$, then $x = w$ +N, where $w \in KM$ and $x = a + N = b + N$, where $a \in A$ and $b \in B$. Thus $w \in A \left( \frac{B}{N} \right)$.

2-Modules with $T$-Pure Intersection Property 
In this section Let $R$ be commutative ring with identity, we introduce the concept of module which have $T$-pure intersection property.

Definition 2.1:
An $R$-module $M$ is said to have the pure relative to submodule intersection property (for short $T$-PIP) if the intersection of any two $T$-pure submodules is again $T$-pure.

Proposition 2.2:
1. If an $R$-module $M$ has the $T$-PIP, then every $T$-pure submodule of $M$ has the $T$-PIP.
2. Let $N$ be $T$-pure submodule of an $R$-module $M$ and $T$ submodule of $N$. $M$ has $T$-PIP if and only if $\frac{M}{N}$ has $\frac{T}{N}$-PIP.

Proof:
1. Clear.
2. ($\Rightarrow$) Conversely let $E$ and $F$ be $T$-pure submodule of $M$, let $N$ be a submodule of $E$ and $N$ be a submodule of $F$ then $\frac{E}{N}$ and $\frac{F}{N}$ is $\frac{T}{N}$-pure submodule of $\frac{M}{N}$. Since $\frac{M}{N}$ has $\frac{T}{N}$-PIP,
then $\frac{E}{N} \cap \frac{F}{N} = \frac{E \cap F}{N}$ is $T$ pure submodule of $\frac{M}{N}$. Therefore $E \cap F$ is $T$- pure submodule of $M$.

**Theorem 2.3:**

Let $M$ be an $R$- module, then $M$ has the T-PIP if and only if

\[(IA \{ B)) + T \{(A \{ B)) \{ IM\} = I( A \{ B) + T \{(A \{ B)) \{ IM\} \text{ for every ideal } I \text{ of } R \text{ and for every } T \text{- pure submodule } A \text{ and } B \text{ of } M.\]

**Proof:**

Suppose $M$ has the T-PIP then for each T-pure submodules $A$ and $B$, $A \{ B$ is T-pure. Let $I$ be an ideal in $R$, then

\[(A \{ B) \{ IM = I (A \{ B) + T \{(A \{ B)) \{ IM\}.\]

It is clear that $I (A \{ B)$ \{ IM \} $\subseteq (IA \{ B) + T \{(A \{ B)) \{ IM\}$. But $IA \{ IB) + T \{(A \{ B)) \{ IM\} \subseteq A \{ B \{ IM = I (A \{ B)$ \{ IM \} .

Thus $IA \{ IB) + J(R) M \{ (A \{ B) \{ IM\} = I (A \{ B) + T \{(A \{ B)) \{ IM\}$.

Conversely, let $A$ and $B$ be T-pure submodule of $M$ and $I$ an ideal in $R$. Then $A \{ B \{ IM = A \{ IB \{ IM \} = A \{ IB \{ IM \}$. Similarly $A \{ B \{ IM = B \{ IM \} = (IA + T \{(B \{ IM\).

But $A, B$ are T- pure in $M$. Thus $A \{ B \{ IM \subseteq IA \{ IB) + T \{(A \{ B) \{ IM\}$.

\[= I (A \{ B) + T (A \{ B) \{ IM)\]

**Theorem 2.4:**

Let $M$ be an $R$- module, then $M$ has the T-PIP if and only if for every T-pure submodules $A$ and $B$ of $M$ and for every $R$- homomorphism $f = A \{ B \rightarrow M$ such that $A \{ Im f) + T \{(A + Im f) \{ IM\} = \{0\}$ and $A + Im f$ is T- pure in $M$, ker $f$ is T- pure in $M$.

**Proof:**

Assume that $M$ has the T-PIP. Let $A$ and $B$ be T-pure submodules of $M$ and $f = A \{ B \rightarrow M$ be an $R$- homomorphism such that $A \{ Im f) + T \{(A + Im f) \{ IM\} = \{0\}$ and $A + Im f$ is T- pure in $M$. Let $K = \{ x + f(x), x \in A \{ B \}$. It is clear that $K$ is a submodule of $M$.

To show that $K$ is T- pure in $M$, let $I$ be an ideal in $R$ and

\[y = \sum_{i=1}^{n} r_{mi} \in K \{ IM, r_{i} \in R, m_{i} \in M.\]

Hence $y = \sum_{i=1}^{n} r_{mi} = x + f(x)$ for some $x \in A \{ B$. Since $y = \sum_{i=1}^{n} r_{mi} = x + f(x) \in A \{ B + Im f$. Let $f \subseteq A + Im f$ and $A + Im f$ is T- pure in $M$.

Thus $y = \sum_{i=1}^{n} r_{mi} \in (A + Im f) \{ IM = I (A + Im f) + T \{(A + Im f) \{ IM\}$.

Therefore

\[\sum_{i=1}^{n} r_{mi} = \sum_{i=1}^{n} r_{xi + yi} + k, x_{i} \in A, y_{i} \in Im f, \forall i = 1, \ldots, n k \in T \{ (A + Im f) \{ IM\}.\]

Thus

\[\sum_{i=1}^{n} r_{mi} = \sum_{i=1}^{n} r_{xi} + \sum_{i=1}^{n} r_{yi} + k, \text{ hence } \sum_{i=1}^{n} r_{xi} = \sum_{i=1}^{n} r_{yi} - f(x) + k \in (A \{ Im f) + T \{(A + Im f) \{ IM\}.\]

Therefore

\[x = \sum_{i=1}^{n} r_{xi} \in (A \{ B) \{ IM\}.\]

But $A \{ B$ is T- pure in $M$, hence is T-pure in $A$. Thus $A \{ B \{ IM = I (A \{ B) + T \{(A \{ B)) \{ IM\}$ by theorem (2.3). Thus $x \in I (A \{ B) + T \{(A \{ B) \{ IM\}$.

Let $x = \sum_{i=1}^{n} r_{wi} + h, w_{i} \in A \{ B, h \in T \{(A \{ B) \{ IM\). Then $f(x) = \sum_{i=1}^{n} rf_{i}(w_{i}) + f(h)$. Now $y = x + f(x) = \sum_{i=1}^{n} r_{wi} + \sum_{i=1}^{n} rf_{i}(w_{i}) + f(h) \in IK + T \{(K \{ IM\). Thus $K \{ IM = IK + T \{(K \{ IM\) and $K$ is T- pure in $M$. Next we show that ker $f = (A \{ B) \{ K$. Let $x \in ker f$, then $x \in A \{ B$ and $f(x) = 0$. Hence $x \in K$.

Now let $x \in (A \{ B) \{ K$, then $x = y + f(y), y \in A \{ B$, then $x - y = f(y) \in A \{ Im f \leq (A \{ B) \{ IM\} = 0$. Therefore
f(x) = f(y) = 0 and x ∈ ker f. Since M has T-PIP, then \( (A \bigcap B) \bigcap K = ker f \) is T-pure in M. Conversely, let A and B be T-pure submodules of M. Define \( f = A \bigcap B \rightarrow M \) by \( f(x) = 0, \forall x \in A \bigcap B \). It is clear \( (A \bigcap IM) + T \bigcap (A + IM) = 0 \) and \( A + IM f = A \) is T-pure in M, then ker f = \( A \bigcap B \) is T-pure in M.

By the same argument one can prove the following

**Theorem 2.5:**

Let M be an R-module, then M has the T-PIP if and only if for every T-pure submodules A and B of M and for every R-homomorphism \( f = A \bigcap B \rightarrow C \), where C is a submodule of M such that \( A \bigcap C + T \bigcap (A + C \bigcap IM) = 0 \) and \( A + C \) is T-pure in M, ker f is T-pure in M.

**References**


