Common Fixed Points for Nonexpansive Mapping by Iteration

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The purpose of this paper is to present convergence iteration scheme which converges strongly in one case and converges weakly in other cases to a common fixed point of a finite set of nonexpansive mappings.

INTRODUCTION AND PRELIMINARIES

There are a number of fixed point theorems that guarantee the existence of a fixed point of iterated sequence (Picard sequence) including the Banach`s fixed point theorem [1], for a contraction mapping defined on a complete metric space.

And then the (Browder- Gohde- Krik) theorem 1965 states that any nonexpansive mapping T on a nonempty closed bounded and convex subset of M an uniformly convex Banach space has at least one fixed point in M, unlike in the case of the Banach`s fixed point theorem, trivial example show that the Picard sequence for nonexpansive mapping T even with a unique fixed point may fail to converge to the fixed point. It suffices, for example, to take T a rotation of unit ball in the plane around the origin of coordinates [2, p.475]. However in this example one can obtain a convergent Picard sequence if instead of T. One take the nonexpansive mapping \( \frac{1}{2}(I+T) \), when I denote the identity mapping of the plane, i.e. if the Picard iteration is defined for \( x_0 \) in M by

\[
x_{n+1} = \frac{1}{2} (x_n + Tx_n), \quad n \geq 0
\]

.. (1.1)
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Instead by the usual Picard iteration. It is easy to see that the mappings \( T \) and \( \frac{1}{2}(I+T) \), have the same fixed points, so the limit of convergent sequence by (1.1) is necessary a fixed point of \( T \).

More generally, if \( X \) is a normed linear space and \( M \) is a convex subset of \( X \) a generalization of (1.1) which has percent successful in the approximation of fixed points of nonexpansive mapping \( T:M \rightarrow M \), is the following scheme [2, p.481],

\[
x_o \in M, \quad x_{n+1} = (1-\alpha)x_n + \alpha T(x_n), \quad n \geq 0, \alpha \in [0,1]. \tag{1.2}
\]

However, the most general iterative

\[
x_o \in M, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(x_n), \quad n \geq 0, \quad \alpha_n \in (0,1), \tag{1.3}
\]

when \( \{\alpha_n\} \subset [0,1] \) is a real sequence satisfying appropriate conditions. The sequence \( \{x_n\} \) in (1.3) is called Mann iteration [3].

**Definition 1.1** [2, p.473] X be a normed space and \( M \) be a nonempty subset of \( X \) A mapping \( T:M \rightarrow X \) is called nonexpansive if:

\[
\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in M \tag{1.4}
\]

**Definition 1.2** [4] Let \( X \) be a Banach space, \( M \) be a nonempty convex subset of \( X \). Suppose \( \{T_i:i=1,2,...,k\} \) is a family of nonexpansive self-mappings of \( M \). Define the following iteration scheme:

\[
U_0 = I, \quad I \text{ the identity map},
\]

\[
U_1 = (1-\alpha)I + \alpha (T_1 o U_0), \quad \alpha \in (0,1),
\]

\[
U_2 = (1-\alpha)I + \alpha (T_2 o U_1),
\]

\[
\begin{align*}
\vdots \\
U_k &= (1-\alpha)I + \alpha (T_k o U_{k-1}).
\end{align*}
\tag{1.5}
\]

\[
x_o \in M, \quad x_{n+1} = (1-\alpha)x_n + \alpha (T_k o U_{k-1})(x_n), \quad n \geq 0. \tag{1.6}
\]

Define \( F = \bigcap_{i=1}^{k} F(T_i) \), where \( F(T_i) \) denotes the fixed point set of \( T_i \).

**Remark 1.3** [4] :Observe that for \( k=1 \), the sequence (1.6) becomes

\[
x_{n+1} = (1-\alpha)x_n + \alpha T_1(x_n), \quad \tag{1.7}
\]

which converges to a fixed point of \( T_1 \) see [5].
Definition 1.4 [6] A normed linear space $X$ is said to be strictly convex when
$$\|x + y\| = \|x\| + \|y\| \iff y = ax \text{ for all } x, y \in X \text{ and } a \geq 0.$$ 

Definition 1.5 [7] Let $X$ be a Banach space, $M$ is a nonempty subset of $X$, $T$ a mapping of $M$ into $X$. Then $T$ is said to be semi contractive if there exists a mapping $V$ of $M \times M$ into $X$ such that $T(u) = V(u, u)$ for $u \in M$, while:

i. For each fixed $v$ in $M$, $V(., v)$ is nonexpansive from $M$ to $X$.
ii. For each fixed $u \in M$, $V(u, .)$ is completely continuous from $M$ to $X$, uniformly for $u$ in bounded subset of $M$ (i.e. if $v_j$ converges weakly to $v$ in $M$ and $\{u_j\}$ is a bounded sequence in $M$, then $V(u_j, v_j) - V(u_j, v) \to 0$, strongly in $M$).

Lemma 1.6 [8] Let $X$ be a uniformly convex Banach space, $M$ be a nonempty closed bounded convex subset of $X$ and $T$ is a semi contractive mapping of $M$ into $X$. Then:

i. $(I - T)$ is demi closed and
ii. $(I - T) M$ is closed in $X$.

Definition 1.7 [2, p.474] A Banach space $X$ is said to be a uniformly convex if and only if for any $x, y, \in X$, $R > 0$ and $\varepsilon \in [0, 2]$, there exists a $\delta(\varepsilon) \in [0, 1]$, such that if $\|x\| \leq R$, $\|y\| \leq R$ and $\|x - y\| \geq \varepsilon R$, then $\frac{1}{2}(x + y) \leq (1 - \delta(\varepsilon)) R$.

Lemma 1.8 [9] Let $M$ a subset of a Banach space $X$ and let $T$ be a nonexpansive mapping from $M$ into $X$. if there exist $x_1$ and $\{a_n\}$ that satisfy condition A and the Mann iteration scheme (1.7) is bounded, then $x_n - T(x_n)$ converges to zero as $n \to \infty$.

Kuhfittig [4], show that Kuhfittting's iteration (1.6) converges strongly to a common fixed point of the family. His second result is that, if $X$ is uniformly convex an satisfies Opail's condition and $M$ is closed convex, then (1.6) converges weakly to a fixed point in $F$. 
Rhoades [10], show that if X is an uniformly convex and M is closed convex then Kuhfittting's iteration (1.6) converges weakly to a fixed point in F.

In this paper, we recall a particular iteration which is Kuhfittting's iteration (1.6) and give all details of two results. The first one due to Kuhfittting who prove that this iteration scheme converges strongly to family of nonexpansive mappings defined on convex compact subset of strictly convex Banach space. The second is Rhoades's results who prove that Kuhfittting's iteration (1.6) converges weakly to a fixed point of a commutative finite of nonexpansive mappings which defined on closed convex subset of uniformly convex Banach space.

**MAIN RESULT**

In order to prove our main theorems we need the following important propositions:

Also Kuhfittig mention to the following without proof

**Proposition 2.1 [4]** Let X be a Banach space and M be a nonempty convex subset of X. Suppose \{T_i: i=1,2,...,k\} is a family of nonexpansive self-mappings on M. Then the mappings \(U_i\) and \(T_i \circ U_i : M \rightarrow M\) for \(i=1,2,...,k\) are nonexpansive mappings.

**Proof:**

\[ U_0 = I, \] identity map and \(U_0(x) = x \) for all \(x \in M\) and \(U_i = (1-\alpha)I + \alpha(T_i \circ U_{i-1})\), then for all \(x, y \in M\)

\[
\| U_1(x) - U_1(y) \| = \| (1-\alpha) x + \alpha(T_1 \circ U_0)(x) - (1-\alpha) y + \alpha(T_1 \circ U_0)(y) \| \\
= (1-\alpha) \| x + \alpha T_1(x) - (1-\alpha) y + \alpha T_1(y) \| \\
\leq (1-\alpha) \| x - y \| + \alpha \| T_1(x) - T_1(y) \| \\
\leq (1-\alpha) \| x - y \| + \alpha \| x - y \| \\
= \| x - y \| \\
\]

Thus \(U_1\) is nonexpansive mapping

Suppose that it is true when \(i=k-1\) (i.e., \(\| U_{k-1}(x) - U_{k-1}(y) \| \leq \| x - y \| \)).

To show that when \(i=k\) (i.e., to prove \(\| U_k(x) - U_k(y) \| \leq \| x - y \| \)).

\[
\| U_k(x) - U_k(y) \| = \| (1-\alpha) x + \alpha(T_k \circ U_{k-1})(x) - (1-\alpha) y + \alpha(T_k \circ U_{k-1})(y) \| \\
= \| (1-\alpha)(x - y) + \alpha(T_k(U_{k-1}(x)) - T_k(U_{k-1}(y))) \| \\
\leq (1-\alpha) \| x - y \| + \alpha \| T_k(U_{k-1}(x)) - T_k(U_{k-1}(y)) \| \\
\]

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\[
\leq (1 - \alpha) \| x - y \| + \alpha \| U_{k-1} (x) - U_{k-1} (y) \|
\leq (1 - \alpha) \| x - y \| + \alpha \| x - y \|
= \| x - y \|
\]
Also, then \( U_i \) is nonexpansive mapping for all \( i \).

Now, for all \( x, y \in M \)
\[
\| (T_i \circ U_i) (x) - (T_i \circ U_i) (y) \| = \| T_i (U_i (x)) - T_i (U_i (y)) \|
\leq \| U_i (x) - U_i (y) \|
\leq \| x - y \|
\]
Thus, then \( T_i \circ U_i \) is nonexpansive for all \( i \).

Demarr [10] proved the following proposition for a commutative family of contractive mappings here we present it with proof for nonexpansive mappings:

**Proposition 2.2** Let \( X \) be a Banach space, \( M \) be a nonempty compact convex subset of \( X \). If \( E \) is a nonempty family of commutative nonexpansive mappings on \( M \), then the family \( E \) has a common fixed point.

For the proved of this proposition we will need the following lemma:

**Lemma 2.3** [11] Let \( M_0 \) be a nonempty convex subset of a Banach space \( X \) and \( f \) is a nonexpansive mapping of \( M_0 \) into itself. If there is a compact set \( K \subseteq M_0 \) such that \( K = f(K) \) and \( K \) has at least two points, then there exists a nonempty closed convex set \( K_1 \) such that \( f(K_1 \cap M_0) \subseteq K_1 \cap M_0 \) and \( K \cap K^*_1 \neq \emptyset \) (\( K^*_1 \) is the complement of \( K_1 \)).

**Proof of the proposition**:

One may show by Zorn's lemma, that there exists a minimal nonempty compact convex set \( M_0 \subseteq M \) such that \( M_0 \) is invariant under each \( f \in E \). If \( M_0 \) consists of a single point, then \( f(M_0) = M_0 \) for each \( f \in E \), implies \( \bigcap_{f \in E} f(M_0) = M_0 \), then the proposition is proved. We shall now show if \( M_0 \) consist of more than one point then we obtain a contradiction.
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We may use Zorn's lemma again to show that there exists a minimal nonempty compact (but not necessarily convex) set $N \subset M_0$ such that $N$ is invariant under each $f \in E$ (i.e., $f(N) \subset N$). We will now show that $N = \{f(x) : x \in N\}$ for each $f \in E$. Since each $f \in E$ is continuous and $N$ is compact, $f(N)$ must also be compact. For all $f \in E$, we have $f(N) \subset N$. Let us assume that for some $g \in E$ we have $g(N) = W \neq N$; therefore for any $x \in W$, there exists $y \in N$ such that $x = g(y)$ because all functions in $E$ is commutative, we have for all $f \in E$, $f(x) = f(g(y)) = g(f(y)) \in W$, $f(x) \in W$.

Thus, we have $f(W) \subset W \subset N$ for each $f \in E$. But $W$ is nonempty compact subset of $M_0$, $f(W) \subset W$ and $W \subset N$ ($W \neq N$). We have contradicted the minimally of $N$. Consequently, our assumption that $N \neq W$ is false. We may assume that $N$ has at least two points; otherwise, the theorem is proved.

We may now apply lemma (2.3) to each $f \in E$. Referring to the notation of lemma (1.5), we see that the set $K_1 \cap M_0$ is invariant under each $f \in E$. Since $K_1$ is closed, we see that $K_1 \cap M_0$ be a nonempty compact convex subset of $M_0$. Since $M_0 \cap K_1 \neq \emptyset$, we see that $K_1 \cap M_0 \neq M_0$. Thus, we see that if $M_0$ has more than one point, then we obtain a contradiction to the minimality of $M_0$.

**Proposition 2.4** Let $X$ be a Banach space and $M$ a convex subset of $X$. If \{T_i : i=1,2,...,k\} is a family of commutative nonexpansive mappings of $M$. Then families \{T_1, T_2,...,T_k\} and \{U_1, U_2,...,U_k\} in (1.5) have the same set of common fixed point.

Proof:

Let $y \in \bigcap_{i=1}^{k} F(T_i)$, then $y \in F(T_i)$ (i.e., $y = T_i(y)$) for all $i$, $U_0(y) = y$.

$$U_1(y) = (1 - \alpha) y + \alpha T_1(y) = (1 - \alpha) y + \alpha y = y$$

Suppose that it is true when $i = k - 1$ (i.e., $U_{k-1}(y) = y$). To prove that when $i = k$ (i.e., to prove $U_k(y) = y$).

$$U_k(y) = (1 - \alpha) y + \alpha (T_k U_{k-1})(y)$$

$$= (1 - \alpha) y + \alpha T_k(U_{k-1}(y))$$

$$= (1 - \alpha) y + \alpha T_k(y)$$
Thus, \( y = U_i(y) \) for all \( i = 1, 2, \ldots, k \).

Here, the finite family of commutative nonexpansive mappings have common fixed points; so, we reform [4, Th. 1] for strongly converges and Rhoades’ theorem for nonexpansive with commutative condition on \( \{T_i: i = 1, 2, \ldots, k\} \).

**Theorem 2.5** Let \( M \) be a nonempty convex compact subset of a strictly convex Banach space \( X \) and \( \{T_i: i = 1, 2, \ldots, k\} \) is a family of commutative nonexpansive self-mappings on \( M \). Then the Mann iterates scheme in (1.6) converges strongly to a common fixed point of \( \{T_i: i = 1, 2, \ldots, k\} \).

**Proof:**

By proposition (2.1) the mappings \( U_i \) and \( T_i o U_{i-1} \), \( i = 1, 2, \ldots, k \) are nonexpansive mappings of \( M \) into itself. By proposition (2.4) the families \( \{T_1, T_2, \ldots, T_k\} \) and \( \{U_1, U_2, \ldots, U_k\} \) have the same set of common fixed point.

Since the sequence (1.6) has the same form as (1.7), \( \{U_k x_n\} \) converges to a fixed point \( y \) of \( T_k o U_{k-1} \) by Edelstein’s theorem [3]. We wish to show next that \( y \) is a common fixed point of \( T_k \) and \( U_{k-1} \) \( (k \geq 2) \). To this end we first show that \( (T_{k-1} o U_{k-1}) (y) = y \) \( (k \geq 2) \). Suppose not; then the closed line segment \([y, (T_{k-1} o U_{k-2}) (y)]\) has positive length. Now let

\[ z = U_{k-1}(y) = (1 - \alpha) y + \alpha (T_{k-1} o U_{k-2}) (y). \]

By hypothesis there exists a point \( w \) such that \( T_1(w) = T_2(w) = \ldots = T_k(w) = w \). Since \( \{T_i\} \) and \( \{U_i\} \) have the same common fixed point, it follows that \( (T_{k-1} o U_{k-2}) (w) = w \). By nonexpansiveness

\[ \| (T_{k-1} o U_{k-2}) (y) - w \| \leq \| y - w \| \] \( \quad \ldots \quad (2.1) \]

and

\[ \| T_k(z) - w \| \leq \| z - w \|. \]

So \( w \) is at least as close to \( T_k(z) \) as to \( z \). But \( T_k(z) = (T_k o U_{k-1}) (y) = y \), so that \( w \) is a least as close to \( y \) as to \( z = (1 - \alpha) y + \alpha (T_{k-1} o U_{k-2})(y) \). Since \( X \) is strictly convex, we conclude that

\[ \| y - w \| < \| (T_{k-1} o U_{k-2}) (y) - w \|. \]

This contradicts (2.5.4), so that \( (T_{k-1} o U_{k-2}) (y) = y \). From

\[ U_{k-1} = (1 - \alpha) I + \alpha (T_{k-1} o U_{k-2}), \]
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follows that $U_{k-1}(y) = (1 - \alpha)y + \alpha y = y$ and $y = (T_k \circ U_{k-1})(y) = T_k(y)$. Consequently, $y$ is a common fixed point of $T_k$ and $U_{k-1}$.

Since $(T_{k-1} \circ U_{k-2})(y) = y$, we may repeat the argument to show that $(T_{k-2} \circ U_{k-3})(y) = y$ and that $y$ must therefore be a common fixed point of $T_{k-1}$ and $U_{k-2}$. Continuing in this manner, we conclude that $(T_{k-2} \circ U_{k-3})(y) = y$ and that $y$ must therefore be a common fixed point of $T_{k-1}$ and $U_{k-2}$. Thus, $y$ is a common fixed point of $\{T_i: i=1,2,\ldots,k\}$.

**Theorem 2.6** Let $X$ be a uniformly convex Banach space, $M$ be a nonempty closed convex subset of $X$ and $\{T_i: i=1,2,\ldots,k\}$ is a family of commutative nonexpansive self-maps on $M$. Then Mann iteration scheme in (1.6) converges weakly to a common fixed point of $\{T_i: i=1,2,\ldots,k\}$.

**Proof:**

By proposition (2.1) the mappings $U_i$ and $T_i \circ U_{i-1}$, $i=1,2,\ldots,k$ are nonexpansive mappings of $M$ into itself. By proposition (2.4) the families $\{T_1, T_2,\ldots,T_k\}$ and $\{U_1, U_2,\ldots,U_k\}$ have the same set of common fixed points. Let $p \in F$, set $S = T_k \circ U_{k-1}$. For any $x \in M$ and $p \in F(T)$, define

$$E = \{u \in X: \|u - p\| \leq r\} \cap M,$$

where $r = \|x - p\|$.

Then $E$ is a nonempty bounded convex subset of $M$ which is invariant under the $U_i$ and $T_i$ and contains $x_0 = x$. Thus, without loss of generality we may assume that $M$ is bounded.

Since $S$ is nonexpansive (1.2),

$$\|x_{n+1} - p\| = \|(1 - \alpha)x_n + \alpha S(x_n)\|$$

$$\leq (1 - \alpha) \|x_n - p\| + \alpha \|S(x_n) - p\|$$

$$\leq (1 - \alpha) \|x_n - p\| + \alpha \|x_n - p\| = \|x_n - p\|$$

Therefore $\lim_{n \to \infty} \|x_n - p\|$ exists, which implies that $\{x_n\}$ is bounded.

From lemma (1.8), $\lim_{n \to \infty} \|x_n - S(x_n)\| = 0$. The assumption that $X$ is uniformly convex implies that it is reflexive. The boundedness of $\{x_n\}$ implies that there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a point $q \in M$. Since $S$ is nonexpansive, if one define $V$ by $V(u,v) = S(u) + v$, then $V$ is semi contractive and from lemma (1.6), $S$ is demi closed. That is, if $\{x_{n_i}\}$ converges weakly to a point $q$, since $\lim_{i \to \infty} \|(I - S)(x_{n_i})\| = 0$, $(I - S)q = q$; so that, $q$
is a fixed point of $S$. A uniformly convex space is strictly convex, so one can use the argument of theorem (2.5), which we now do, to show that $q \in F(T)$.

Suppose that $q$ is not a common fixed point of $T_{k-1}$ and $U_{k-2}$. Then the closed line segment $[q, (T_{k-1} \circ U_{k-2})q]$ has positive length. Define

$$z = U_{k-1}(q) = (1-\alpha)q + \alpha (T_{k-1} \circ U_{k-2})(q).$$

By hypothesis there exists a point $w$ such that $T_1(w) = T_2(w) = \ldots = T_k(w) = w$. Since $\{T_i\}$ and $\{U_i\}$ have the same common fixed point, it follows that

$$(T_{k-1} \circ U_{k-2})(w) = w.$$  

Since $\{T_i\}$ and $\{U_i\}$ are nonexpansive

$$\| (T_{k-1} \circ U_{k-2})(q) - w \| \leq \| q - w \| \quad \ldots \quad (2.2)$$

$$\| T_k(z) - w \| \leq \| z - w \|.$$

Therefore $w$ is at least as close to $T_k(z)$ as to $z$. But $T_k(z) = (T_k \circ U_{k-1})(q) = q$, so that $w$ is at least as close to $q$ as to $z = (1-\alpha)q + \alpha (T_{k-1} \circ U_{k-2})(q)$. Since $X$ is strictly convex, it follows that

$$\| q - w \| < \| (T_{k-1} \circ U_{k-2})(q) - w \|.$$  

This contradicts (2.2), so that $(T_{k-1} \circ U_{k-2})(q) = q$. It now follows from $U_{k-1} = (1-\alpha)I + \alpha (T_{k-1} \circ U_{k-2})$ that $U_{k-1}(q) = (1-\alpha)q + \alpha q = q$ and $q = (T_k \circ U_{k-1})(q) = T_k(q)$. Therefore, $q$ is a common fixed point of $T_k$ and $U_{k-1}$.

Since $(T_{k-1} \circ U_{k-2})(q) = q$, we may repeat the above argument to show that $(T_{k-2} \circ U_{k-3})(q) = q$ and that $q$ must therefore be a common fixed point of $T_{k-1}$ and $U_{k-2}$. Continuing in this manner, we conclude that $(T_1 \circ U_0)(q) = q$ and that $q$ is a common fixed point of $T_2$ and $U_1$. Thus, $q$ is a common fixed point of $\{T_i: i=1,2,\ldots,k\}$.

**Corollary 2.7** [4, Th.3] If $X$ is a uniformly convex Banach space satisfying Opial’s condition, $M$ is a nonempty closed convex subset of $X$ and if the family of mappings $\{T_i: i=1,2,\ldots,k\}$ satisfies (1.2), then for any $x \in M$, the sequence $\{x_n\}$ converges weakly to a common fixed point.

**Proof:**

It is clear.

**REFERENCES**


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