Stability analysis in a discrete-time predator-prey model with allee effect

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ABSTRACT

In this paper, a discrete-time predator-prey model involving allee effect is investigated. The existence and local stability of all possible fixed points are carried out. It is observed that, the allee effect (phenomenon in biology refer to the positive relationship between aspects of fitness and population size) made very interesting system when it imposed in predator population.

INTRODUCTION

The dynamical relationship between predator and its prey is continuing to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. There are plenty of papers about the dynamics on the predator-prey system with and without different kinds of functional responses, so it is worth mentioning that the consequences of hiding behavior of prey on the dynamics of predator-prey interactions can be recognized significant [2].

In fact, the Allee effect is a phenomenon in biology named after allee [3] who brought an attention to the possibility of a positive relationship between aspects of fitness and population size over fifty years ago [4]. In the other words, for smaller population, the reproduction and survival of individuals decrease. This effect usually saturates or disappears as populations get larger. For more details see [5-11] and the reference within.

Merdan and Duman[12] presented the stability of the discrete-time model involving predation and allee effect in general and showed that an allee
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effect has stabilizing role in there model. Furthermore, Celik et al [13] studied a discrete-time predator-prey with and without allee effect; they imposed the effect of Allee on the prey population and exhibit the impact of allee effect on stability.

Throughout this paper, we consider the discrete-time model with allee effect in the closed first quadrant $R^+_2$ of the $(x,y)$ plane. Also, we discuss the existence of fixed points and studied the stability of it by calculating eigenvalues for the variation matrix at each fixed point. Furthermore, the predator population occurs with absence of prey population so it is fixed point found and studied too.

The aim of the present work is to propose the discrete-time predator-prey model involving allee effect in predator population. It is organized as follow: in the section 2, the discrete-time predator-prey model with allee effect is formulated. In section 3, the existence and local stability conditions of each fixed points are investigated. In section 4, some numerical simulations are presented. Finally, in section 5, the discussion and remarks are drawn.

The model:

The discrete-time predator-prey model is described by difference equations be in the following form:

\[
\begin{align*}
X_{n+1} &= X_n + aX_n (1 - X_n) - bX_n Y_n \\
Y_{n+1} &= Y_n + bY_n (X_n - Y_n)
\end{align*}
\]

(I)

Where $a,b > 0$ such that $a$ is the parameter of the increases of the prey population in the absence of predator and $b$ is the parameter of the decrease due to predation. While $X_n$ and $Y_n$ represent the densities of the prey and predator populations at the iteration $n(n=0,1,...)$, respectively.

Now, consider the system (I) as a subject to an allee effect on predator population. Then we get the following system:
\[\begin{align*}
X_{n+1} &= X_n + aX_n(l - X_n) - bX_n Y_n \\
Y_{n+1} &= Y_n(l - bY_n) \frac{Y_n}{\varepsilon + Y_n} + bX_n Y_n
\end{align*}\]

(2)

Where we take \(\frac{Y_n}{\varepsilon + Y_n}\) as an allee function and \(\varepsilon\) as an allee constant that satisfying the assumption:

\[
\frac{(ab - b^2 \varepsilon) \pm \sqrt{(ab - b^2 \varepsilon)^2 - (ab + b^2)(a\varepsilon - ab\varepsilon)}}{2(ab + b^2)} > 0
\]

(3)

**Remark 2.1:**

If the predator density disappears in the systems (1) and (2), then the prey population satisfies the logistic equation and vice versa. While if \(\varepsilon = 0\) in the system (2), then there is no allee effect on the predator population.

**The existence and stability of fixed points:**

In this section, we first determine the existence of the fixed points of the system (2), and then investigate their stability by calculating the eigenvalues for the variation matrix of the system (2) at each fixed point.

By solving the following nonlinear algebraic equations

\[
\begin{align*}
X(a(l - X) - bY) &= 0 \\
Y(\frac{Y}{\varepsilon + Y}(l - bY) + bX - 1) &= 0
\end{align*}
\]

(4)

We get the fixed points \((0,0), (1,0), (0, \frac{l}{b})\) and \((X^*, Y^*)\) where \(X^* = \frac{a - bY^*}{a}\) and \(Y^* = \frac{(ab - b^2 \varepsilon) \pm \sqrt{(ab - b^2 \varepsilon)^2 - (ab + b^2)(a\varepsilon - ab\varepsilon)}}{2(ab + b^2)}\), obviously \((0, \frac{l}{b})\) exist if \(b > 0\) and \((X^*, Y^*)\) are the only positive fixed points which exist if \(a > bY^*\) and \(Y^*\) have positive real values.

To study the stability of the fixed points of our modification model, we first recall the useful lemma, which can be easily proved by the relation between roots and coefficients of a quadratic equation [14].
Lemma 3.1 [14]:

Let \( F(\lambda) = \lambda^2 - B\lambda + C \). Suppose that \( F(1) > 0 \) then the two roots \( \lambda_1 \) and \( \lambda_2 \) of \( F(\lambda) = 0 \) satisfy the following:

(i) \( |\lambda_1| < 1 \) and \( |\lambda_2| < 1 \) if and only if \( F(-1) > 0 \) and \( C < 1 \);
(ii) \( |\lambda_1| > 1 \) and \( |\lambda_2| > 1 \) if and only if \( F(-1) > 0 \) and \( C > 1 \);
(iii) \( |\lambda_1| < 1 \) and \( |\lambda_2| > 1 \) (or \( |\lambda_1| > 1 \) and \( |\lambda_2| < 1 \)) if and only if \( F(-1) < 0 \);
(iv) \( \lambda_1 = -1 \) and \( |\lambda_2| \neq 1 \) if and only if \( F(-1) = 0 \) and \( B \neq 0, 2 \);
(v) \( \lambda_1 \) and \( \lambda_2 \) are complex and \( |\lambda_1| = |\lambda_2| = 1 \) if and only if \( B^2 - 4C < 0 \) and \( C = 1 \).

Now, let \( \lambda_1 \) and \( \lambda_2 \) be the two eigenvalues of the fixed point \((x,y)\). We recall some definitions of topological types for a fixed point \((x,y)\). A fixed point \((x,y)\) is called a sink if \( |\lambda_1| < 1 \) and \( |\lambda_2| < 1 \), so the sink is locally asymptotically stable. \((x,y)\) is called a source if \( |\lambda_1| > 1 \) and \( |\lambda_2| > 1 \), so the source is locally unstable. \((x,y)\) is called a saddle if \( |\lambda_1| < 1 \) and \( |\lambda_2| > 1 \) (or \( |\lambda_1| > 1 \) and \( |\lambda_2| < 1 \)). And \((x,y)\) is called non-hyperbolic if either \( |\lambda_1| = 1 \) or \( |\lambda_2| = 1 \).

Substituting the coordinates of the fixed point \((0,0)\) for \((X,Y)\) of the system (2) and computing the eigenvalues of the fixed point \((0,0)\), so we obtained the following proposition.

**Proposition 3.2:** The origin fixed point \((0,0)\) of the system (2) is a saddle.

**Proof:**

The Jacobian matrix of the system (2) at the fixed point \((0,0)\) is given as follow

\[
J_o = \begin{bmatrix}
1 + a & 0 \\
0 & 0 \\
\end{bmatrix}
\]  

and the corresponding characteristic equation can be written as

\[
\lambda^2 - (trJ_o)\lambda + det(J_o) = 0
\]  

Where \( trJ_o \) is the trace and \( det(J_o) \) is the determinant of the Jacobian matrix \( J_o \). Hence the two eigenvalues of the Jacobian matrix \( J_o \) are \( \lambda_1 = 1 + a > 1 \) and \( \lambda_2 = 0 < 1 \). According to the lemma 1, we obtained that there is only one topological type of the origin fixed point \((0,0)\) for all parameters values, which means it is a saddle fixed point.
Proposition 3.3: The axial fixed point (1,0) has at least four different topological types, that means (1,0) is:

i. Sink if \( a < 2 \) and \( b < 1 \);
ii. Source if \( a > 2 \) and \( b > 1 \);
iii. Non-hyperbolic if \( a = 2 \) or \( b = 1 \);
iv. Saddle otherwise.

From the condition (iii) of the proposition 3.3, it is easy to see that one of the eigenvalues of the fixed point (1,0) is \(-1\) and the other eigenvalue is neither 1 nor \(-1\) and then it also implies that all the parameters locate in the following set:
\[
H_{(1,0)} = \{(a,b): a \neq 2, b = 1\}.
\]

The fixed point (1,0) can undergo flip bifurcation when parameters vary in the small neighborhood of \( H_{(1,0)} \), since when parameters are in \( H_{(1,0)} \) [15]. In this case, the predator becomes extinction and the prey pass through the period-doubling bifurcation to chaos in the sense of Li-Yorke by choosing bifurcation.

Proposition 3.4: if \( b > 0 \), then the fixed \((0, \frac{1}{b})\) satisfying the following topological types for all permissible values of parameters.

i. Sink if \( a < 1 \) and \( be > 0 \),
ii. Non-hyperbolic if \( a = 1 \) or \( be = 0 \)

The eigenvalues of the matrix \( J_2 \), are \( \lambda_1 = a \) and \( \lambda_2 = -\frac{1}{1+be} \). So, from the existence condition of the fixed point \((0, \frac{1}{b})\) we have that \( b > 0 \) then we can say that the second eigenvalue depend on the allee constant and the fixed point \((0, \frac{1}{b})\) is non-hyperbolic if there is no allee effect or \( a = 1 \).

Now, we recall the well known condition [16] and are sufficient for the local stability of the positive fixed point. The fixed point \((X*, Y*)\) is stable if it satisfies the following conditions:
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\[
\begin{aligned}
1 + Tr(J_3) + Det(J_3) > 0 \\
1 - Tr(J_3) + Det(J_3) > 0 \\
1 - Det(J_3) > 0 
\end{aligned}
\]

(10)

**Proposition 3.5:** By assumption (3), the positive fixed point \((X^*, Y^*)\) of the system (2) is asymptotically stable if the following conditions are satisfied:

i. \(\gamma > 1\)

ii. \[
\frac{(a - bY^*)(a + a\beta - b^2Y^*)}{a(\beta + 1)} < 2 \quad \text{and} \quad \beta \neq -1
\]

iii. \[
\frac{(a - bY^*)(a\beta - b^2Y^*)}{a(\beta - 1)} > 1 \quad \text{and} \quad \beta \neq 1
\]

.\text{where} \(\beta = \frac{b(a - bY^*) + \varepsilon Y^*(1 - bY^*) + Y^*(1 - 2bY^*)}{aY^* (\varepsilon + (1 - 3\varepsilon)Y^* - 2bY^*^2)}\), \(\gamma = \frac{(\varepsilon + Y^*^2)(a - ab + 2b^2Y^*)}{aY^*[2\varepsilon + (1 - 3\varepsilon)Y^* - 2bY^*^2]}\).

So such that \(2\varepsilon + (1 - 3\varepsilon)Y^* - 2bY^*^2 \neq 0\).

**Proof:**

After some calculation, the Jacobian matrix of the system (2) at the fixed point \((X^*, Y^*)\) is:

\[
J_3 = \begin{bmatrix}
1 - a + bY^* & -\frac{b(a - bY^*)}{a} \\
bY^* & \beta
\end{bmatrix}
\]

(11)

So the characteristic equation is:

\[
\lambda^2 - (1 - a + bY^* + \beta)\lambda + \beta(1 - a + bY^*) + \frac{b^2Y^*(a - bY^*)}{a} = 0
\]

(12)

Where

\[
Tr(J_3) = 1 - a + bY^* + \beta
\]

(13)

and

\[
Det(J_3) = \beta(1 - a + bY^*) + \frac{b^2Y^*(a - bY^*)}{a}
\]

(14)
From the conditions (9), we obtained that the modulus of two roots of the equation (11) are in the unit circle if and only if $F(I) > 0$, $F(-I) > 0$ and $Det(J_3) < 1$. Now, according to lemma 1, we observed that $F(I) > 0$ holds if and only if $\gamma > 1$. Then we investigate the conditions $F(-I) > 0$ and $Det(J_3) < 1$ when $\gamma > 1$. It implies that $F(-I) > 0$ holds if and only if $\frac{(a-b\gamma)x + a\gamma - b\gamma^2}{a(\beta+\gamma)} < 2$ holds, and $Det(J_3) < 1$ holds if and only if $\frac{(a-b\gamma)x + a\gamma - b\gamma^2}{a(\beta+\gamma)} > 1$ holds, so the proof is complete.

**Numerical simulation:**

In this section, we give the numerical simulations to verify our theoretical results proved in the previous sections. Mainly, we present the graph of the solutions $X_n$ versus $Y_n$ for the prey-predator model with and without allee effect (systems (1) and (2), respectively) and show the impact of the allee effect on the phase portrait of the solutions.

Likely, çelik et al [13] fixed parameter values satisfying the existence conditions of the positive fixed point. So, we used it in our numerical simulations.

When we analyze the phase portrait of the solutions around the positive fixed point for both systems, we can easily see the stabilizing effect of allee function that we imposed on predator population by system (2).

In figure (1) we illustrate the phase portrait of prey and predator densities in systems (1) and (2) by taking $b = 2$ and the initial values $x_0 = 0.3, y_0 = 0.2$. We used $a = 1.4$ in figures 1 (a) and 1 (c) while $a = 2.2$ in figures 1 (b) and 1 (d). Here (a) and (b) show the phase portrait of prey and predator densities in system (1), however (c) and (d) correspond to system (2) taking the same parameters as in (a) and (b).

We see from figures 1 (a) and 1 (c) that in system (1), the local stability of fixed point and trajectory of the solution approximates to the corresponding fixed point much faster than system (2). Furthermore, figures 1 (b) and 1 (d) presents that the corresponding fixed point move from period-4 to chaotic under the allee effect. Here we take the allee effect function as $\frac{Y_n}{\varepsilon + Y_n}$ and take the allee constant as $\varepsilon = 0.09$. 


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**Fig. 1:** The graph in (a) (resp. (b)) indicates the solution of system (1) with $a=1.4$ (resp. $a=2.2$), however, the graph in (c) (resp. (d)) corresponds to system (2) with $\epsilon=0.09$.

Figure (2) shows the bifurcation diagrams of prey and predator densities of systems (1) and (2) with the initial values $x_0=0.3, y_0=0.2$ as above and the parameter values $b=2, \epsilon=0.09$ and $a$ in the $[1.93]$ with step size $=0.01$. 

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In figure (2), we can see that the stability region in figures 2 (a) and 2 (b) are as same as in figures 2 (c) and 2 (d) while the instability region gives rich dynamic and interesting one when the predator densities is subject to the allee effect with allee constant as $\varepsilon = 0.09$.

**Fig.-2: Bifurcation diagrams of prey and predator densities with growth rate in systems (1) and (2)**

**Discussion:**
In this paper, the discrete –time prey model with an allee effect was proposed. Existence and stability of fixed points were investigated by...
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mathematical analysis; we have shown the stability of the positive fixed point. However, it may be very complicated structure when our system delayed both prey and predator populations as subject to an allee effect. Thus it would be very interesting to improve such structure in the future.

REFERENCES