Approximate Solution Of The Linear Mixed Volterra-Fredholm Integro Differential Equations Of Second kind
By Using Variational iteration Method

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ABSTRACT
In this work, the variatioal iteration method (VIM) is employed to finding the approximation solution of the linear mixed Volterra-Fredholm integro differential equation of second kind (V-FIEs). The (VIM) is to construct correction functional using general Lagrange multipliers(\(\lambda\)) identified optimally via the variational theory. We proving theorem study the convergence approximate solutions to the exact solutions, Finlly ,two examples are given and their results are given in tables and are shown in figures, the error estimate ,in each examples is calculated.

INTRODUCTION
A mixed Volterra- Fredholm integro differential equation contain mixed Volterra and Fredholm integral equations where the Fredholm integral is the interior integral ,whereas the volterra integral is the exterior one.Moreover , the unknown function \(u(x,y)\) appears inside the integral ,whereas the derivative \(\frac{\partial u}{\partial y}(x,y)\) appears outside the integral.

These types of equations playing an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of elasticity, engineering and mathematical physics[1,2,3,4]. A discussion of the formulation of these models is given in wazwaz[4] and the references therein.

In this work, we consider the following the linear mixed Volterra-Fredholm differential equation of second kind:

\[
\frac{\partial u}{\partial y}(x,y) = g(x,y) + \int_0^x \int_{\Omega} k(x,y,z,m)u(z,m)dz dm \quad (x,y) \in [0, T] \times \Omega
\]

where \(u(x,y)\) is an unknown function, the known functions \(g(x,y)\) and \(k(x,y,z,m)\) are continuous of x and y on \(D=[0,T]\times\Omega\) and \((S\times\mathbb{R})\) where \(S=\{(x,y,z,m),0\leq z, m \leq T,(x,y)\subset\Omega\times\Omega\}\) and \((\Omega\text{ is a closed subset of }\mathbb{R}^n (n=1,2,3,...))\).
The existences and uniqueness to Eq.(1) are given in [5]. Many research are used the linear mixed Volterra-Fredholm integral and integro differential equation by Brunner H.[6], Shazad S.[7] and wazwaz A. [4]. In this work we propose procedure for solving the linear mixed Volterra-Fredholm integro equation of second kind using variational iteration method.

**Variational Iteration Method:**

The variational iteration method was proposed by Ji-Huan He in 1999[8,9], and yet powerful method for solving a wide class of linear and nonlinear problems. The (VIM) gives rapidly convergent successive approximation of the exact solution if such a solution exists[9]. This method is based on use of lagrange multipliers for identification of optimal value of a parameter in a functional [10, 11].

To illustrate the basic idea of the (VIM), we consider the following general functional equation given in operator form:

\[ L(u(x,y)) + N(u(x,y)) = g(x,y) \]  

where \( L \) is a linear operator. \( N \) is a nonlinear operator and \( g(x,y) \) is any given function which is called the nonhomogeneous term. According to the (VIM), we can construct the following correction functional

\[ u_{n+1}(x, y) = u_n(x, y) + \int_0^y \lambda \{ L(x, y) + N\tilde{u}(x, y) - g(x, y) \} \]

where \( \lambda \) is a general lagrange multiplier which can be identified optimally via variation theory[10,11,12], \( u_0 \) is an initial approximation in this work knowns functions, and \( \tilde{u} \) consider as restricted variation which means \( \delta \tilde{u} = 0 \). Therefore, we first determine the lagrange multiplier \( \lambda \) that will be \( u_{n+1}(x, y) \geq 0 \) of the solution \( u(x, y) \) will be readily obtained upon using the lagrange multiplier obtained by using any selective function \( u_0(x, y) \), consequently the solution \( u(x, y) = \lim_{n \to \infty} u_n(x, y) \).

**Variational Iteration Method Solving The Linear Mixed Volterra-Fredholm Integro Differential Equation Of Second Kind (V-FIEs):**

Now, we consider the linear mixed Volterra-Fredholm integro differential equation of of second kind:

\[ \frac{\partial u}{\partial y}(x, y) = g(x, y) + \int_a^b \int_a^x k(x, y, z, m)u(z, m)dz dm \]  

Then we have the following iteration sequence

\[ u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda(\xi)\{ u_n(\xi, y) - g(\xi, y) \} - \int_a^x \int_a^b k(\xi, y, z, m)u_n(\xi, m)dz dm d\xi \]  

\[ \lambda = \frac{\partial u}{\partial y}(x, y) - g(x, y) \]
To find the optimal $\lambda$, we proceed as follows:

$$\delta u_{n+1}(x,y) = \delta u_n(x,y) + \int_0^y \lambda(\xi)\left\{\delta u_n(\xi,y) \cdot g(\xi,y) - \int_a^b k(\xi,y,z,m)u_n(\xi,m) \, dz \, dm\right\} d\xi = 0$$

... (5)

and upon using the method of integration by parts, then Eq.(5) will be reduced to:

$$\delta u_{n+1}(x,y) = \delta u_n(x,y) + \int_0^y \lambda(\xi)\left\{\delta u_n(\xi,y) + \lambda(x,y)\delta u_n(x,y) + \int_a^x \hat{\lambda}(\xi)\delta u_n(\xi,y) = 0\right\}$$

Then the following stationary conditions are obtained:

$$\hat{\lambda} = 0, \quad \lambda + 1 = 0$$

The general Lagrange multipliers therefor, can be readily identified: $\lambda = -1$ and by substituting in Eq.(4), the following iteration formula $n \geq 0$ is obtained

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^y \left\{\delta u_n(\xi,y) \cdot g(\xi,y) - \int_a^x \int_a^b k(\xi,y,z,m)u_n(\xi,m) \, dz \, dm\right\} d\xi$$

... (6)

**Theorem:** Let $u \in (C^2[a, b], || \cdot ||_\infty)$ be the exact solution of the linear mixed Volterra-Fredholm integro differential equation of (3) and $u_n \in C^2[a, b]$ be the obtained solution of the sequence defined by Eq.(4). If $E_n(x) = u_n(x,y) - u(x,y)$ and $|k| \leq c, \quad 0 < c < 1$, then the sequence of approximate solutions $\{u_n\}, n = 0, 1, \ldots$ converges to the exact solution $u(x,y)$.

**Proof:**
Consider the linear mixed Volterra-Fredholm integro differential equation of second:

$$\frac{\partial}{\partial y} u(x,y) = g(x,y) + \int_a^x \int_a^b K(x,y,z,m)u(z,m) \, dz \, dm$$

Where the approximate solution using the VIM is given by

$$u_{n+1}(x,y) = u_n(x,y) - \int_0^y \left[\frac{\partial}{\partial s} u_n(x,s) - g(x,s) - \int_a^x \int_a^b k(x,s,z,m)u_n(z,m) \, dz \, dm\right] ds$$

...(7)

And since $u$ is exact solution of the linear mixed VFIEs, have

$$u(x,y) = u(x,y) - \int_0^y \left[\frac{\partial}{\partial s} u(x,s) - g(x,s) - \int_a^x \int_a^b k(x,s,z,m)u(z,m) \, dz \, dm\right] ds \quad \ldots (8)$$
Now, subtracting Eq.(8) from Eq.(7) to get:

\[ E_{n+1}(x,y) = \]
\[ E_n(x,y) - \int_0^y \int_a^b k(x,s,z,m)E_n(z,m)dzdm ds \]
\[ E_{n+1}(x,y) = E_n(x,y) - \int_0^y \frac{\partial}{\partial s} E_n(x,s)ds \]
\[ = E_n(x,y) - \int_0^y \int_a^b k(x,s,z,m)E_n(z,m)dzdm ds \]

And since \( E_n(x,0) = u_n(x,0) - u(x,0) \), which have the initial condition of the VFIEs, the \( E_n(x,0) = 0 \). Hence

\[ E_{n+1}(x,y) = \int_0^y \int_a^b k(x,s,z,m)E_n(z,m)dzdm \quad \cdots(9) \]

Now, taking the maximum-norm on both sides of Eq.(9), yields to:

\[ \|E_{n+1}(x,y)\|_\infty = \left\| \int_0^y \int_a^b k(x,s,z,m)E_n(z,m)dzdm \right\|_\infty \]
\[ \|E_{n+1}(x,y)\|_\infty \leq \int_0^y \int_a^b \|k\|_\infty \|E_n(z,m)\|_\infty dzdm \]

since \( K \) is function bounded by \( c, c \in (0,1) \), then

\[ \|E_{n+1}(x,y)\|_\infty \leq c \int_0^y \int_a^b \|k\|_\infty \|E_n(z,m)\|_\infty dzdm \]
\[ \|E_{n+1}(x,y)\|_\infty = cy \int_a^b \|E_n(z,m)\|_\infty dzm \]

Therefore

\[ \|E_{n+1}(x,y)\|_\infty \leq cy \int_a^b \|E_n(z,m)\|_\infty dzdm, \forall n = 0,1, \cdots(10) \]

Now, if \( n=0 \), then inequality (10) yield to

\[ \|E_1(x,y)\|_\infty \leq cy \int_a^b \|E_0(z,m)\|_\infty dzdm \]
\[ \|E_1(x,y)\|_\infty \leq cy \int_a^b \frac{\text{Max}}{(z,m)}E_0(z,m)dzdm \]
\[ \|E_1(x,y)\|_\infty = cy(x-a)(b-a)\text{Max}|E_0| \]

(11)

Also, if \( n=1 \), then from inequality (10) and (11) we have

\[ \|E_2(x,y)\|_\infty \leq cy \int_a^b \|E_1(z,m)\|_\infty dzdm \]

Substituting (11), in this inequality we get

\[ \|E_2(x,y)\|_\infty \leq cy \int_a^b cy(x-a)(b-a)\text{Max}|E_0| \]
\[ \|E_2(x,y)\|_\infty = c^2y^2 \frac{(x-a)^2}{2} (b-a)^2 \text{Max}|E_0| \]

(12)

Similarly, for \( n=2 \) and from inequality (10) and (12), we have

\[ \|E_3(x,y)\|_\infty \leq cy \int_a^b \|E_2(z,m)\|_\infty dzdm \]

Substituting (12), in this inequality we get

\[ \|E_3(x,y)\|_\infty \leq cy \int_a^b c^2y^2 \frac{(x-a)^2}{2} (b-a)^2 \text{Max}|E_0| dzdm \]
\[ \|E_3(x,y)\|_\infty = c^3y^3 \frac{(x-a)^3}{3!} (b-a)^3 \text{Max}|E_0| \]

And so on, in general and using mathematical induction we get:

\[ \|E_n(x,y)\|_\infty \leq c^n y^n \frac{(x-a)^n}{n!} (b-a)^n \text{Max}|E_0| \]

(13)
And since $c \in (0,1)$ and as $n \to \infty$, then we will have the right hand side of inequality (13) tends to zero, i.e., $\|E_n(x,y)\|_\infty \to 0$ as $n \to \infty$

Which implies to $u_n(x,y) \to u(x,y)$ as $n \to \infty$

i.e., the sequence of solutions obtaind from the VIM converge of the exact solution $u(x,y)$.

**Numerical Examples:**

In the section, we used the (VIM) which is discussed of the previous section for solve two examples.
Example (1):
Consider the linear mixed Volterra-Fredholm integro differential equation of second kind

\[ \frac{\partial}{\partial y} u(x,y) = x - \frac{1}{10} x^3 y + \int_0^1 \int_0^1 z^3 x y u(z,m) dz \, dm \]

With exact solution \( u(x,y) = xy \)

The corresponding iterative formula (6) for this example can be constructed as follows:

\[ u_{n+1}(x,y) = u_n(x,y) - \int \left\{ \hat{u}_n(\xi,y) - g(\xi,y) - \int_0^b \int_a^b k(\xi,y,z,m) u_n(\xi,m) \, dz \, dm \right\} \, d\xi \]

Let the initial approximate solution \( u_0(x,y) = x - \frac{1}{10} y x^3 \), we get:

\[ u_1(x,y) = u_0(x,y) - \int \left\{ u_0(x,y) - (x - \frac{1}{10} y x^3) - \int_0^1 \int_0^1 z^3 x y u_0(z,m) \, dz \, dm \right\} \, d\xi \]

\[ = \frac{1}{240} [-12y^2 x^3 - yx^3 + 20ytx^2 + 240yx + 240x] \]

\[ u_2(x,y) = u_1(x,y) - \int \left\{ u_1(x,y) - (x - \frac{1}{10} x^3 y) - \int_0^1 \int_0^1 z^3 x y u_1(z,m) \, dz \, dm \right\} \, d\xi \]

\[ = \frac{1}{240} [-12y^2 x^3 - yx^3 + 20ytx^2 + 240yx + 240x] - \frac{1}{40320} [63x^4 y - 2561] \]

And so on, we may compute \( u_0, u_{11}, u_{13} \); which be more complicated.

The exact solution and the approximate solution \( u_N \) with different \( N \) and the absolute error \( |e_N| = |u_{\text{exact}} - u_N| \), Table(1) and Figure(1) & (2) are shown exact and approximate solutions by using variational iteration Method.
Table-1: The results of Example(1)

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<th>Approximate Sol. at N=10</th>
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L.S.E  3.38e-08  1.968e-011  7.53e-15  4.75e-022

Figure-1: the exact solution
Figure-2: The approximate solution
Example (2):-
Consider the Linear mixed Volterra-Fredholm integro differential equation of second kind

\[ u'(x, y) = 2ye^x - (x^2e^{-1}e^x - x^2e^{-1})(e^2 - 5) + \int_0^x \int_{-1}^1 x^2e^{-z} u(z, m) dz \, dm \]

With exact solution \( u(x,y) = y^2e^x \)

The corresponding iterative formula (6) for this example can be constructed as follows:

\[ u_{n+1}(x, y) = u_n(x, y) - \int_0^x \int_a^b \left\{ u_{n}(\xi, y) - g(\xi, y) - \int_0^x \int_{-1}^1 x^2e^{-z} u_n(\xi, m) \, dz \, dm \right\} d\xi \]

Let the initial approximate solution \( u_0(x, y) = 2ye^x - (x^2e^{-1})(e^2 - 5) \)
we get:

\[ u_1(x, y) = u_0(x, y) - \int_0^y \left\{ u_0'(x, y) - (2ye^x - (x^2e^{-1})(e^2 - 5)) - \int_0^x \int_{-1}^1 x^2e^{-z} u_0(z, m) \, dz \right\} \]

\[ = y^2e^x - 0.8788x^2e^x + 0.8788x^2y - 0.58592yx^3 + 2yx^4 + 0.8788x^2 - 4.3944yx^3e^{-1} + 0.8788yx^3e^{-1} - 0.8788yx^2e^x \]

\[ u_2(x, y) = u_1(x, y) - \int_0^y u_1'(x, y)(2ye^x - (x^2e^{-1})(e^2 - 5)) - \int_0^x \int_{-1}^1 x^2e^{-z} u_1(z, m) \, dz \]

\[ = y^2e^x - 0.8788x^2e^x + 0.8788x^2y - 0.58592yx^3 + 2yx^4 + 0.8788x^2 - 4.3944yx^3e^{-1} + 0.8788yx^3e^{-1} - 0.8788yx^2e^x + xye^{-2}(0.89e^3 + 13.625e^4 + 0.666e^2 + 35.155) \]

The exact solution and the approximate solution \( u_N \) with different \( N \) and the absolute error \( |e_N| = |u_{\text{exact}} - u_N| \). Table(1) and Figure(3) & (4) are shown exact and approximate solutions by using variational iteration Method.
Table-2: The results of Example(2)

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<td>2.85e-011</td>
<td></td>
<td>1.35e-015</td>
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Figure-3: Approximate solution using VIM
Figure-4: Exact solution

CONCLUSION
In this paper the variational iteration method is used to solve the linear mixed Volterra-Fredholm integro differential equation with second kind. The results showed that the convergence and accuracy of variational iteration method for numerically solution for (V-FIE) were in a good
agreement with analytical solutions. The computations associated with examples and graphing in this paper performed using matlab (v 6.5).

REFERENCES


