On (1,2)* b-Open Functions and (1,2)* b-Closed Functions In Bitopological Spaces

Sabiha I. Mahmood* and Sanaa Hamdi**

*Department of Mathematics, College of Science, Al-Mustansiryah University, Baghdad-Iraq.
**Department of Mathematics, College of Education, Al-Mustansiriyah University, Baghdad-Iraq.

Abstract

The main goal of this paper is to create special type of open and closed functions in bitopological spaces namely, quasi (1,2)* b-open functions and quasi (1,2)* b-closed functions. Also, we give some properties and equivalent statements of this concept.

Keywords: (1,2)* b-continuous function, (1,2)* b-irresolute function, contra (1,2)* b-irresolute function, quasi (1,2)* b-open function and quasi (1,2)* b-closed function.

Introduction

The concept of a bitopological space $(X, \tau_1, \tau_2)$ was first introduced by Kelly [1], where $X$ is a nonempty set and $\tau_1, \tau_2$ are topologies on $X$. Also, the concept of $(1,2)*$ b-open sets was first introduced and studied by Sreeja and Janaki [2]. The purpose of this paper is to give a new type of open and closed functions in bitopological spaces called quasi $(1,2)*$ b-open functions and quasi $(1,2)*$ b-closed functions. Also, we study the relation between the quasi $(1,2)*$ b-open (resp. quasi $(1,2)*$ b-closed) functions and each of the $(1,2)*$ open (resp. $(1,2)*$ closed) functions, $(1,2)*$ b-open (resp. $(1,2)*$ b-closed) functions and pre-$(1,2)*$ b-open (resp. pre-$(1,2)*$ b-closed) functions. Moreover, we study the characterizations and basic properties of quasi $(1,2)*$ b-open functions and quasi $(1,2)*$ b-closed functions.

Throughout this paper $(X, \tau_1, \tau_2)$, $(Y, \sigma_1, \sigma_2)$ and $(Z, \eta_1, \eta_2)$ (or simply $X, Y$ and $Z$) represent non-empty bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned.

1. Preliminaries

First, we recall the following definitions:

(1.1) Definition [3]:
A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$-open if $A=U_1\cup U_2$ where $U_1 \in \tau_1$ and $U_2 \in \tau_2$. The complement of a $\tau_1\tau_2$-open set is called $\tau_1\tau_2$-closed.

Notice that $\tau_1\tau_2$-open sets need not necessarily form a topology [3].

(1.2) Definition [3]:
Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A \subseteq X$. Then:-

i) The $\tau_1\tau_2$-closure of $A$, denoted by $\tau_1\tau_2c(A)$, is defined by:
$$\tau_1\tau_2c(A)=\{F: A \subseteq F \& F \text{ is } \tau_1\tau_2 \text{ closed}\}$$

ii) The $\tau_1\tau_2$-interior of $A$, denoted by $\tau_1\tau_2int(A)$, is defined by:
$$\tau_1\tau_2int(A)=\{U: U \subseteq A \& U \text{ is } \tau_1\tau_2 \text{ open}\}.$$

(1.3) Definition [4]:
A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called an $(1,2)*$-neighborhood of a point $x$ in $X$ if there exists a $\tau_1\tau_2$-open set $U$ in $X$ such that $x \in U \subseteq A$.

(1.4) Definition [2]:
A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(1,2)*$ b-open if $A\subseteq \tau_1\tau_2d(cl(\tau_1\tau_2int(A)) \cap \tau_1\tau_2int(cl(\tau_1\tau_2d(A)))$. The complement of an $(1,2)*$ b-open set is said to be $(1,2)*$ b-closed. The class of all $(1,2)*$ b-open (resp. $(1,2)*$ b-closed) subsets of $X$ is denoted by $(1,2)*BQX(\tau_1, \tau_2)$ (resp. $(1,2)*BQX(\tau_1, \tau_2)$).

(1.5) Definition:
Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A \subseteq X$. Then :-

i) The $(1,2)*$ b-closure of $A$, denoted by $(1,2)*bc(A)$, is defined by:
$$\{F: A \subseteq F \& F \text{ is } (1,2)*b \text{--closed}\}.$$
ii) The \((1,2)^*\) b-interior of \(A\), denoted by \((1,2)^*\text{bint}(A)\) is defined by: \((1,2)^*\text{bint}(A) = U[U: U \subseteq A \& U \text{ is } (1,2)^*\text{b--open}].\)

The following proposition holds. The proof is easy and hence omitted.

\(1.6\) Proposition:
Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A \subseteq X\). Then:
1) The union (resp. intersection) of any family of \((1,2)^*\) b-open (resp. \((1,2)^*\) b-closed) sets in a bitopological space \((X, \tau_1, \tau_2)\) is \((1,2)^*\) b-open (resp. \((1,2)^*\) b-closed).
2) \(A \subseteq (1,2)^*\text{bc}(A)\).
3) \((1,2)^*\text{bc}(A)\) is an \((1,2)^*\) b-closed set in \(X\).
4) \(A\) is \((1,2)^*\) b-closed in \(X\) iff \(A = (1,2)^*\text{bc}(A)\).
5) \((1,2)^*\text{bint}(A) \subseteq A\).
6) \((1,2)^*\text{bint}(A)\) is an \((1,2)^*\) b-open set in \(X\).
7) \(A\) is \((1,2)^*\) b-open iff \(A = (1,2)^*\text{bint}(A)\).

\(1.7\) Definition [5]:
A function \(f:\ (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is said to be \((1,2)^*\) -continuous if \(f^{-1}(V)\) is \(\tau_1\tau_2\)-open set in \(X\) for every \(\sigma_1\sigma_2\)-open set \(V\) in \(Y\).

\(1.8\) Definition [6]:
A function \(f:\ (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is said to be \((1,2)^*\)-open (resp. \((1,2)^*\)-closed) if the image of every \(\tau_1\tau_2\)-open (resp. \(\tau_1\tau_2\)-closed) subset of \(X\) is a \(\sigma_1\sigma_2\)-open (resp. \(\sigma_1\sigma_2\)-closed) set in \(Y\).

\(1.9\) Definition [2]:
A function \(f:\ (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is said to be \((1,2)^*\) b-irresolute if \(f^{-1}(V)\) is \((1,2)^*\) b-open set in \(X\) for every \((1,2)^*\) b-open set \(V\) in \(Y\).

\(1.10\) Proposition:
A function \(f:\ (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is \((1,2)^*\) b-irresolute iff \(f^{-1}(V)\) is \((1,2)^*\) b-closed set in \(X\) for every \((1,2)^*\) b-closed set \(V\) in \(Y\).

\(1.11\) Definition [7]:
A function \(f:\ (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is said to be pre-(1,2)* b-closed (resp. pre-(1,2)* b-open) if the image of every \((1,2)^*\) b-closed (resp. \((1,2)^*\) b-open) subset of \(X\) is \((1,2)^*\) b-closed (resp. \((1,2)^*\) b-open) set in \(Y\).

\(2.1\) Definition:
A function \(f:\ (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is said to be \((1,2)^*\) b-open if the image of every \(\tau_1\tau_2\)-open subset of \(X\) is \((1,2)^*\) b-open set in \(Y\).

\(2.2\) Definition:
A function \(f:\ (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is said to be quasi \((1,2)^*\) b-open if the image of every \((1,2)^*\) b-open set in \(X\) is \(\sigma_1\sigma_2\)-open in \(Y\).

\(2.3\) Proposition:
Every quasi \((1,2)^*\) b-open function is \((1,2)^*\)-open as well as \((1,2)^*\) b-open.

\(2.4\) Remark:
The converse of (2.3) may not be true in general. Consider the following example.

Example:
Let \(X = Y = \{a,b,c\}, \tau_1 = \{X, \emptyset, \{a\}\}, \tau_2 = \{X, \emptyset\}\), \(\sigma_1 = \{Y, \emptyset, \{a\}, \{b,c\}\}\) \& \(\sigma_2 = \{Y, \emptyset\}\). So the sets in \(\{X, \emptyset, \{a\}\}\) are \(\tau_1\tau_2\)-open in \(X\) and the sets in \(\{Y, \emptyset, \{a\}, \{b,c\}\}\) are \(\sigma_1\sigma_2\)-open in \(Y\). Also, \((1,2)^*\text{BQX}_{\tau_1, \tau_2} = \{X, \emptyset, \{a,c\}, \{a,b\}, \{\emptyset\}\}\) & \((1,2)^*\text{BQY}_{\sigma_1, \sigma_2} = \{Y, \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}\).

Let \(f:\ (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be a function defined by: \(f(a) = a, f(b) = b \& f(c) = c\). It is clear that \(f\) is \((1,2)^*\) b-open as well as \((1,2)^*\)-open, but \(f\) is not quasi \((1,2)^*\) b-open, since \(\{a, c\}\) is \((1,2)^*\) b-open in \((X, \tau_1, \tau_2)\), but \(f(\{a,c\}) = \{a, c\}\) is not \(\sigma_1\sigma_2\)-open in \((Y, \sigma_1, \sigma_2)\).

\(2.5\) Proposition:
Every quasi \((1,2)^*\) b-open function is pre-(1,2)* b-open.

\(2.6\) Remark:
The converse of (2.5) may not be true in general. Consider the following example.
Example:
Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi\}$, $\tau_2 = \{X, \phi\} \cup \{\{a\}, \{b\}, \{a, b\}\}$, $\sigma_1 = \{Y, \phi\} \cup \{\{a\}, \{b\}, \{a, b\}\}$ and $\sigma_2 = \{Y, \phi\} \cup \{\{a\}\}$. So the sets in $\{X, \phi\} \cup \{\{a\}, \{b\}, \{a, b\}\}$ are $\tau_1 \tau_2$-open in $X$ and the sets in $\{Y, \phi\} \cup \{\{a\}, \{b\}, \{a, b\}\}$ are $\sigma_1 \sigma_2$-open in $Y$.

Also, $(1,2)^* B(X, \tau_1, \tau_2) = \{X, \phi\} \cup \{\{a\}, \{b\}, \{a, b\}\}$ and $(1,2)^* BQ(X, \sigma_1, \sigma_2) = \{Y, \phi\} \cup \{\{a\}, \{b\}, \{a, b\}\}$.

Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function defined by: $f(a) = a$, $f(b) = b$, and $f(c) = c$. It is clear that $f$ is pre-$(1,2)^*$-open, but $f$ is not quasi $(1,2)^*$-open, since $\{a, b\}$ is $(1,2)^*$-open in $(X, \tau_1, \tau_2)$, but $f(\{a, b\}) = \{a, b\}$ is not $\sigma_1 \sigma_2$-open in $(Y, \sigma_1, \sigma_2)$.

Thus we have the following diagram:

$\text{quasi } (1,2)^* \text{-open } \Rightarrow \text{pre- } (1,2)^* \text{-open } \Rightarrow \text{(1,2)^* -open } \Rightarrow (1,2)^* \text{-b-open }$

(2.7) Theorem:
A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is quasi $(1,2)^*$-b-open iff $f((1,2)^* \text{bint}(U)) \subseteq \sigma_1 \sigma_2 \text{int}(f(U))$ for every subset $U$ of $X$.

Proof: $\Rightarrow$
Let $f$ be a quasi $(1,2)^*$-b-open function. To prove that $f((1,2)^* \text{bint}(U)) \subseteq \sigma_1 \sigma_2 \text{int}(f(U))$ for every subset $U$ of $X$.

By (1.6) no. 5, $(1,2)^* \text{bint}(U) \subseteq U \Rightarrow f((1,2)^* \text{bint}(U)) \subseteq f(U)$.

Since $(1,2)^* \text{bint}(U)$ is an $(1,2)^*$-open set in $X$ and $f$ is quasi $(1,2)^*$-b-open, then $f((1,2)^* \text{bint}(U))$ is $\sigma_1 \sigma_2$-open in $Y$. Thus $f((1,2)^* \text{bint}(U)) \subseteq \sigma_1 \sigma_2 \text{int}(f(U))$.

Conversely, suppose that $f((1,2)^* \text{bint}(U)) \subseteq \sigma_1 \sigma_2 \text{int}(f(U))$ for every subset $U$ of $X$. To prove that $f$ is quasi $(1,2)^*$-b-open. Let $U$ be an $(1,2)^*$-b-open set in $X$. Then by (1.6) no. 7, $U = (1,2)^* \text{bint}(U) \Rightarrow f(U) = f((1,2)^* \text{bint}(U)) \subseteq \sigma_1 \sigma_2 \text{int}(f(U))$.

But $\sigma_1 \sigma_2 \text{int}(f(U)) \subseteq f(U)$. Consequently $\sigma_1 \sigma_2 \text{int}(f(U)) = f(U) \Rightarrow f(U) \Rightarrow f(U)$ is a $\sigma_1 \sigma_2$-open set in $Y$. Hence $f$ is a quasi $(1,2)^*$-b-open function.

(2.8) Theorem:
If a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is quasi $(1,2)^*$-b-open, then $(1,2)^* \text{bint}(f^{-1}(U)) \subseteq f^{-1}(\sigma_1 \sigma_2 \text{int}(U))$ for every subset $U$ of $Y$.

Proof:
Let $U$ be any arbitrary subset of $Y$. Then $f^{-1}(U)$ is a subset of $X$. Since $f$ is quasi $(1,2)^*$-b-open, then by (2.7) $f((1,2)^* \text{bint}(f^{-1}(U))) \subseteq \sigma_1 \sigma_2 \text{int}(f^{-1}(U)) \subseteq \sigma_1 \sigma_2 \text{int}(U)$.

Thus $(1,2)^* \text{bint}(f^{-1}(U)) \subseteq f^{-1}(\sigma_1 \sigma_2 \text{int}(U))$ for every subset $U$ of $Y$.

(2.9) Definition:
A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be a $(1,2)^*$ b-neighborhood of a point $x$ in $X$ if there exists an $(1,2)^*$ b-open set $U$ in $X$ such that $x \in U \subseteq A$.

(2.10) Theorem:
Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function from a bitopological space $X$ into a bitopological space $Y$. Then the following are equivalent:

i) $f$ is quasi $(1,2)^*$-b-open.
ii) $f((1,2)^* \text{bint}(U)) \subseteq \sigma_1 \sigma_2 \text{int}(f(U))$ for each subset $U$ of $X$.
iii) for each $x \in X$ and each $(1,2)^*$-b-neighborhood $U$ of $x$ in $X$, there exists an $(1,2)^*$-neighborhood $V$ of $f(x)$ in $Y$ such that $V \subseteq f(U)$.

Proof:
(i) $\rightarrow$ (ii). It follows from theorem (2.7).
(ii) $\rightarrow$ (iii).

Let $x \in X$ and $U$ be an arbitrary $(1,2)^*$ b-neighborhood of $x$ in $X$, then by (2.9) there exists an $(1,2)^*$ b-open set $V$ in $X$ such that $x \in V \subseteq U$. Since $V$ is $(1,2)^*$ b-open in $X$, then by (1.6) no. 7, $V = (1,2)^* \text{bint}(V)$. By (ii), we have $f(V) = f((1,2)^* \text{bint}(V)) \subseteq \sigma_1 \sigma_2 \text{int}(f(V)) \Rightarrow f(V) \subseteq \sigma_1 \sigma_2 \text{int}(f(V))$.

Since $\sigma_1 \sigma_2 \text{int}(f(V)) \subseteq f(U) \Rightarrow (1,2)^* \text{bint}(f^{-1}(U)) \subseteq f^{-1}(\sigma_1 \sigma_2 \text{int}(U))$ for every subset $U$ of $Y$. 

\[\text{Thus we have the following diagram:}\]

\[\text{quasi } (1,2)^* \text{-open } \Rightarrow \text{pre- } (1,2)^* \text{-open } \Rightarrow \text{(1,2)^* -open } \Rightarrow (1,2)^* \text{-b-open }\]
f(V)=σ₁σ₂ intf(V)⇒ f(V) is σ₁σ₂-open in Y such that f(x)∈f(V)⊆f(U).

(iii)→(i).

Let U be an arbitrary (1,2)* b-open set in X. Then for each y∈f(U) there exists x∈U such that f(x)=y. By (iii) there exists an (1,2)*-neighborhood V of y in Y such that V⊂f(U). Since V is an (1,2)*-neighborhood of y, then there exists a σ₁σ₂-open set W_y in Y such that y∈W_y⊂V. Thus f(U)=∪_{y∈f(U)} W_y which is a σ₁σ₂-open set in Y. This implies that f is quasi (1,2)* b-open function.

(2.11) Theorem:
A function f:(X,τ₁,τ₂)→(Y,σ₁,σ₂) is quasi (1,2)* b-open iff for any subset B of Y and for any (1,2)* b-closed set F of X containing f⁻¹(B), there exists a σ₁σ₂-closed set G of Y containing B such that f⁻¹(G)⊂F.

Proof: ⇒
Suppose that f is quasi (1,2)* b-open. Let B⊂Y and F be an (1,2)* b-closed subset of X such that f⁻¹(B)⊂F. Now, put G=Y−f(X−F). Since f⁻¹(B)⊂F ⇒ X−F⊂f⁻¹(B) ⇒ f(X−F)⊂f(f⁻¹(B))⊂B ⇒ B⊂Y−f(X−F) ⇒ B⊂G. Since f is quasi (1,2)* b-open, then G is a σ₁σ₂-closed subset of Y. Moreover, we have f⁻¹(G)⊂F.

Conversely, let U be an (1,2)* b-open set in X. To prove that f(U) is a σ₁σ₂-open set in Y. Put B=Y−f(U), then X−U is an (1,2)* b-closed set in X such that f⁻¹(B)⊂X−U. By hypothesis, there exists a σ₁σ₂-closed subset F of Y such that B⊂F and f⁻¹(F)⊂X−U. Hence, we obtain f(U)⊂Y−F. On the other hand, since B⊂F ⇒ Y−F⊂Y−B=f(U) ⇒ Y−F⊂f(U). Thus f(U)=Y−F which is a σ₁σ₂-open and hence f is a quasi (1,2)* b-open function.

(2.12) Theorem:
A function f:(X,τ₁,τ₂)→(Y,σ₁,σ₂) is quasi (1,2)* b-open iff f⁻¹(σ₁σ₂c(B))⊂(1,2)*bc(f⁻¹(B)) for every subset B of Y.

Proof: ⇒
Suppose that f is quasi (1,2)* b-open. To prove that f⁻¹(σ₁σ₂c(B))⊂(1,2)*bc(f⁻¹(B)) for every subset B of Y. Since f⁻¹(B)⊂(1,2)*bc(f⁻¹(B)) for any subset B of Y, then by (2.11) there exists a σ₁σ₂-closed set F in Y such that B⊂F and f⁻¹(F)⊂(1,2)*bc(f⁻¹(B)). Since B⊂F ⇒ σ₁σ₂c(B)⊂σ₁σ₂c(F). Therefore, we obtain f⁻¹(σ₁σ₂c(B))⊂f⁻¹(F)⊂(1,2)*bc(f⁻¹(B)). Thus f⁻¹(σ₁σ₂c(B))⊂(1,2)*bc(f⁻¹(B)) for every subset B of Y.

Conversely, let B⊂Y and F be an (1,2)* b-closed subset of X such that f⁻¹(B)⊂F. Put W=σ₁σ₂c(B), then we have B⊂W and f⁻¹(W)=f⁻¹(σ₁σ₂c(B))⊂(1,2)*bc(f⁻¹(B))⊂(1,2)*bc(F)=F. Then by theorem (2.11) f is a quasi (1,2)* b-open function.

However the following theorem holds. The proof is easy and hence omitted.

(2.13) Theorem:
Let f:(X,τ₁,τ₂)→(Y,σ₁,σ₂) and g:(Y,σ₁,σ₂)→(Z,η₁,η₂) be two functions. Then:-
1) If f and g are quasi (1,2)* b-open, then g◦f is quasi (1,2)* b-open.
2) If f and g are quasi (1,2)* b-open, then g◦f is quasi (1,2)* b-open.
3) If f is quasi (1,2)* b-open and g is quasi (1,2)* b-open, then g◦f is quasi (1,2)* b-open.
4) If f is quasi (1,2)* b-open and g is quasi (1,2)* b-open, then g◦f is quasi (1,2)* b-open.
5) If f is quasi (1,2)* b-open and g is quasi (1,2)* b-open, then g◦f is quasi (1,2)* b-open.
6) If f is quasi (1,2)* b-open and g is quasi (1,2)* b-open, then g◦f is quasi (1,2)* b-open.
7) If f is quasi (1,2)* b-open and g is quasi (1,2)* b-open, then g◦f is quasi (1,2)* b-open.
8) If \( f \) is \((1,2)^*\)-open and \( g \) is quasi \((1,2)^*\) b-open, then \( g \circ f \) is \((1,2)^*\)-open.

2.14) Definition:

A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is said to be \((1,2)^*\) b-continuous if \( f^{-1}(V) \) is \((1,2)^*\) b-open set in \( X \) for every \( \sigma_1 \sigma_2 \)-open set \( V \) in \( Y \).

2.15) Proposition:

A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((1,2)^*\) b-continuous iff \( f^{-1}(V) \) is \((1,2)^*\) b-closed set in \( X \) for every \( \sigma_1 \sigma_2 \)-closed set \( V \) in \( Y \).

2.16) Definition:

A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is said to be contra \((1,2)^*\) b-irresolute if \( f^{-1}(V) \) is \((1,2)^*\) b-closed set in \( X \) for every \( \sigma_1 \sigma_2 \)-open set \( V \) in \( Y \).

2.17) Theorem:

Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) and \( g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) be two functions. Then:-

1) If \( g \circ f \) is quasi \((1,2)^*\) b-open and \( g \) is \((1,2)^*\)-continuous and one-to-one, then \( f \) is quasi \((1,2)^*\) b-open.

2) If \( g \circ f \) is quasi \((1,2)^*\) b-open and \( g \) is \((1,2)^*\)-continuous and one-to-one, then \( f \) is pre-(\((1,2)^*\) b-open.

3) If \( g \circ f \) is contra \((1,2)^*\) b-irresolute and \( g \) is quasi \((1,2)^*\) b-open and one-to-one, then \( f \) is contra \((1,2)^*\) b-irresolute.

4) If \( g \circ f \) is quasi \((1,2)^*\) b-open and \( f \) is \((1,2)^*\) b-irresolute and onto, then \( g \) is \((1,2)^*\)-open.

Proof:

1) To prove that \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is quasi \((1,2)^*\) b-open. Let \( U \) be an \((1,2)^*\) b-open subset of \( X \), since \( g \circ f \) is quasi \((1,2)^*\) b-open, then \((g \circ f)(U) \) is \( \eta_1 \eta_2 \)-open in \( Z \). Since \( g \) is \((1,2)^*\)-continuous, then \( g^{-1}(g \circ f(U)) = (g^{-1} \circ g)(f(U)) \) is \( \sigma_1 \sigma_2 \)-open in \( Y \). Since \( g \) is one-to-one, then \( f(U) \) is \( \sigma_1 \sigma_2 \)-open in \( Y \). Thus \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is a quasi \((1,2)^*\) b-open function.

2) To prove that \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pre-(\((1,2)^*\) b-open. Let \( U \) be an \((1,2)^*\) b-open subset of \( X \), since \( g \circ f \) is quasi \((1,2)^*\) b-open, then \((g \circ f)(U) \) is \( \eta_1 \eta_2 \)-open in \( Z \). Since \( g \) is \((1,2)^*\)-continuous, then \( g^{-1}(g \circ f(U)) = (g^{-1} \circ g)(f(U)) \) is \( \sigma_1 \sigma_2 \)-open in \( Y \). Since \( g \) is one-to-one, then \( f(U) \) is \( \sigma_1 \sigma_2 \)-open in \( Y \). Thus \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is a pre-(\((1,2)^*\) b-open function.

3. Quasi \((1,2)^*\) b-closed Functions

3.1) Definition:

A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is said to be \((1,2)^*\) b-closed if the image of every \( \tau_1 \tau_2 \)-closed subset of \( X \) is \((1,2)^*\) b-closed set in \( Y \).

3.2) Definition:

A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is said to be quasi \((1,2)^*\) b-closed if the image of
(3.3) Proposition:
Every quasi (1,2)* b-closed function is (1,2)*-closed as well as (1,2)* b-closed.

(3.4) Remark:
The converse of (3.3) may not be true in general. Consider the following example.

Example:
Let \( X = Y = \{a, b, c\} \) & \( \tau_1 = \{X, \phi, \{a, c\}\} \), \( \tau_2 = \{X, \phi\} \), \( \sigma_1 = \{Y, \phi, \{a, c\}\} \) & \( \sigma_2 = \{Y, \phi, \{a\}\} \). So the sets in \( \{X, \phi, \{b\}\} \) are \( \tau_1 \tau_2 \)-closed in \( X \) and the sets in \( \{Y, \phi, \{b, c\}\} \) are \( \sigma_1 \sigma_2 \)-closed in \( Y \).

Also, \( (1,2)^*BC(X, \tau_1, \tau_2) = \{X, \phi, \{a, \}, \{b, \}, \{c, \}, \{a, \}, \{b, \}, \{c, \}\} \) & \( (1,2)^*BC(Y, \sigma_1, \sigma_2) = \{Y, \phi, \{b, \}, \{c, \}\} \).

Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a function defined by: \( f(a) = a, f(b) = b \) & \( f(c) = c \). It is clear that \( f \) is \( (1,2)^* \) b-closed as well as \( (1,2)^* \)-closed, but \( f \) is not quasi \( (1,2)^* \) b-closed, since \( \{a\} \) is \( (1,2)^* \) b-closed in \( X \), but \( f(\{a\}) = \{a\} \) is not \( \sigma_1 \sigma_2 \)-closed in \( Y \).

Thus we have the following diagram:

\[
\text{quasi (1,2)* b-closed} \quad \Downarrow \quad \text{pre-(1,2)* b-closed} \quad \Downarrow \\
(1,2)^* - \text{closed} \quad \Rightarrow \quad (1,2)^* \text{- b-closed}
\]

(3.7) Theorem:
A bijective function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is quasi \( (1,2)^* \) b-closed iff it is quasi \( (1,2)^* \) b-open.

Proof:
Let \( f \) be a quasi \( (1,2)^* \) b-closed function. To prove that \( f \) is a quasi \( (1,2)^* \) b-open function. Let \( U \) be an \( (1,2)^* \) b-open set in \( X \Rightarrow \exists U^c \subseteq (1,2)^* \) b-closed in \( X \). Since \( f \) is quasi \( (1,2)^* \) b-closed, then \( f(U^c) \) is \( \sigma_1 \sigma_2 \)-closed in \( Y \). Therefore \( f(U^c)^c \) is \( \sigma_1 \sigma_2 \)-open in \( Y \). Since \( f \) is a bijective function, then \( f(U^c)^c \) =f(U) \Rightarrow f(U) is \( \sigma_1 \sigma_2 \)-open in \( Y \). Thus \( f : X \rightarrow Y \) is a quasi \( (1,2)^* \) b-open function.

Conversely, Suppose that \( f : X \rightarrow Y \) is quasi \( (1,2)^* \) b-open. To prove that \( f \) is quasi \( (1,2)^* \) b-closed. Let \( F \) be an \( (1,2)^* \) b-closed set in \( X \Rightarrow \exists F \subseteq (1,2)^* \) b-closed in \( X \). Since \( f \) is quasi \( (1,2)^* \) b-open, then \( f(F) \) is \( \sigma_1 \sigma_2 \)-open in \( Y \). Therefore \( f(F)^c \) is \( \sigma_1 \sigma_2 \)-closed in \( Y \). Since \( f \) is a bijective function, then \( f(F)^c \) =f(F) \Rightarrow f(F) is \( \sigma_1 \sigma_2 \)-closed in \( Y \). Thus \( f : X \rightarrow Y \) is a quasi \( (1,2)^* \) b-closed function.

(3.8) Theorem:
A function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) from a bitopological space \( X \) into a bitopological space \( Y \) is quasi \( (1,2)^* \) b-closed iff \( \sigma_1 \sigma_2 \mathcal{C}(f(F)) \subseteq f((1,2)^*bc(F)) \) for every subset \( F \) of \( X \).

Proof:
\( \Rightarrow \) Let \( f \) be a quasi \( (1,2)^* \) b-closed function. To prove that \( \sigma_1 \sigma_2 \mathcal{C}(f(F)) \subseteq f((1,2)^*bc(F)) \) for every subset \( F \) of \( X \). By (1.6) no.2, \( F \subseteq (1,2)^*bc(F) \Rightarrow f(F) \subseteq f((1,2)^*bc(F)) \).

Since \( (1,2)^*bc(F) \) is an \( (1,2)^* \) b-closed set in \( X \) and \( f \) is quasi \( (1,2)^* \) b-closed, then \( f((1,2)^*bc(F)) \) is \( \sigma_1 \sigma_2 \)-closed in \( Y \). Thus \( \sigma_1 \sigma_2 \mathcal{C}(f(F)) \subseteq f((1,2)^*bc(F)) \) for every subset \( F \) of \( X \).

Conversely:
Suppose that \( \sigma_1 \sigma_2 \mathcal{C}(f(F)) \subseteq f((1,2)^*bc(F)) \) for every subset \( F \) of \( X \). To prove that \( f \) is quasi \( (1,2)^* \) b-closed. Let \( F \) be an \( (1,2)^* \) b-closed set in \( X \). Then by (1.6) no. 4, \( F=(1,2)^*bc(F) \Rightarrow f(F)=f((1,2)^*bc(F)) \).

By hypothesis \( \sigma_1 \sigma_2 \mathcal{C}(f(F)) \subseteq f((1,2)^*bc(F)) \) \( \Rightarrow \) \( \sigma_1 \sigma_2 \mathcal{C}(f(F)) \subseteq f((1,2)^*bc(F)) \Rightarrow f(F) \).

But \( f(F) \subseteq \sigma_1 \sigma_2 \mathcal{C}(f(F)) \). Consequently \( f(F)=\sigma_1 \sigma_2 \mathcal{C}(f(F)) \Rightarrow f(F) \) is a \( \sigma_1 \sigma_2 \)-
closed set in Y. Thus f is a quasi (1,2)* b-closed function.

(3.9) Theorem:
A function \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is quasi (1,2)* b-closed iff for any subset B of Y and for any (1,2)* b-open set G of X containing \( f^{-1}(B) \), there exists a \( \sigma_1\sigma_2 \)-open set U of Y containing B such that \( f^{-1}(U) \subseteq G \).

Proof: 
This proof is similar to that of theorem (2.11).

However the following theorem holds. The proof is easy and hence omitted.

(3.10) Theorem:
Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) and \( g: (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) be two functions. Then:
1) If f and g are quasi (1,2)* b-closed, then \( g \circ f \) is quasi (1,2)* b-closed.
2) If f and g are quasi (1,2)* b-closed, then \( g \circ f \) is pre-(1,2)* b-closed.
3) If f is quasi (1,2)* b-closed and g is (1,2)*-closed, then \( g \circ f \) is quasi (1,2)* b-closed.
4) If f is quasi (1,2)* b-closed and g is (1,2)*-closed and onto, then g is (1,2)*-closed.
5) If \( g \circ f \) is contra (1,2)* b-irresolute and f is quasi (1,2)* b-closed and onto, then g is contra (1,2)* b-irresolute.

Proof:
This proof is similar to that of theorem (2.17).

References