**Vector groupoid and Isometries**

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**ABSTRACT**

The concept of groupoid considers as a generalization of the concept of group, and the notion of an action of groupoid generalizes the action of group relative to a bundle. In this paper we present the concept of groupoid into linear algebra using a structure of vector space, this groupoid namely "vector groupoid". The purpose of this paper is to investigate many properties of vector groupoid comparable with that satisfy for groupoid in algebraic case; also, we study these properties in a topological case using the concept of topological vector groupoid.

**Key words :** Groupoid, Vector bundle, Isometries, Action of top. group and top. groupoid

**INTRODUCTION**

The concept of groupoid is one of the means by which the twentieth century reclaims the original domain of application of group concept. A groupoid is a powerful and flexible generalization of the notion of the group.

In 1920's, Brandt and Baer gave the algebraic theory of groupoid. In 1950's, Ehresmann introduced the groupoid into the differential geometry. Also, the concept of groupoid action is due originally to Ehresmann (1959), generalizing the group action in his work on fiber spaces.

In this paper we present a new type of groupoids called "vector groupoid" $(G,B)$ which represent a generalization of the concept of
vector space, and by adding topological structure on this type of groupoids we get that \((x, x) \in G, \beta \in B\) is a topological vector bundle for each \(x \in B\) with a nice conditions that satisfied on \(G\). Using the regular action of a special type of topological groups we define special kind of topological vector groupoids (Ehresmann vector groupoid \(E \times E / \Gamma\)) which acts regularly (principally and linearly) on \((E, \pi, E / \Gamma)\). Also, we introduce new concept of groupoid actions * Isometric action of a topological groupoid on a finite-dimensional topological vector bundle * and then we link this action of Ehresmann (vector) groupoid \(E \times E / \Gamma\) with the regular action of a special type of topological groups.

1) Groupoid

(1) Algebraic case

(1.1.1) \([1][2]\) A groupoid is a pair of sets \((G, B)\) on which are given , the source and the target mappings ( resp.) \(\alpha, \beta : G \to B\), the mapping of unities \(\omega : B \to G\), the partial composition mapping \(\mu : G \ast G \to G\) and the inversion mapping \(\nu : G \to G\). The mappings \(\alpha, \beta\) defines:

* The subsets of \(G\), \(\alpha(x) \in G\) called the \(\alpha\)-fiber at \(x \in B\), \(\beta(y) \in G\) called the \(\beta\)-fiber at \(y \in B\), \(\alpha(x) \cup \beta(y)\) called the \(\alpha, \beta\)-fiber at \(x, y \in B\), \(\alpha(x) \cap \beta(y)\) the set of all elements in \(G\) with source "\(x\)" and target "\(y\)" \((g : x \to y)\). The subset \(x \times G\) is a group with the restriction of \(\mu\) on \(x \times G \times G\) called vertex(isotropy) group at \(x \in B\) with unity \(\omega(x) \in G\).

*The mapping \(\tau\) \((\beta \times \alpha) \circ \Delta : G \to B \times B\) defined by \(\tau(g) = (\beta(g), \alpha(g)) \forall g \in G\) is called the transitor of \(G\) ( where \(\Delta\) is the diagonal mapping of \(G\) ). A groupoid \((G, B)\) is said to be transitive if the transitor \(\tau\) of \(G\) is surjection.

(2) Topological case

(1.2.1) \([3][4]\) A topological groupoid is a groupoid \((G, B)\) together with topologies on \(G\) and \(B\) such that the mappings \(\alpha, \omega\), partial composition \(\mu\) and
inversion law \( \nu \) are continuous. From this def. we have that \( \alpha \) and \( \beta \) are identification mappings, the mapping of unities \( \omega \) is a topological embedding and the inversion law \( \nu \) is a homeomorphism.

*For any topological groupoid \((G, B)\) and for each \(x \in B\), \(G\) is a topological group. The mapping \((\cdot): G \times G \to G; (g, h) \mapsto gh^{-1}\) is an isomorphism of topological groups, and the mappings \(: g \mapsto LG \to G\) (left translation); \(ha \mapsto Rg \to G\) (right translation); \(hahg\) are homeomorphisms for each \(y, g \in G\). If \(G\) is a transitive groupoid, we have an isomorphism class of topological groups \(x \in B\).

*A morphism of topological groupoids is a morphism of groupoids \((f, f):(G, B) \to (G', B')\), such that \(f\) is continuous. Denote by \(TG\) the category of topological groupoids and their morphisms, and \(T\) the category of topological spaces and continuous mappings. Notice that \((\cdot): (\cdot)(\cdot) \circ \tau I \to B \times B\) is a morphism in \(TG\) where \(\tau\) is the transitor of \(G\) [1]. An isomorphism of topological groupoids is a morphism \((f, f)\) such that \(f\) is a homeomorphism.

(1.2.2) Let \(\phi:E \times \Gamma \to E\) be a law of a continuous (right) action of a topological group \(\Gamma\) on a topological space \(E\), then \(\phi\) defines two types of topological groupoids \(E \times \Gamma\),

\[E \times E\] (the fiber product of the canonical mapping \(\pi_1 : E \to B\) \(E / \Gamma\) by itself over \(B\)) called action groupoids which they are not in general isomorphic in \(TG\) [4].

*A continuous free action \(\phi:E \times \Gamma \to E\) is called principal if \(E \times \Gamma\) and

\[E \times E\] are isomorphic in \(TG\). A principal action of a topological group \(\Gamma\) on a topological space \(E\) defines a morphism:

\[\phi \to \Gamma \to E; r \mapsto z \cdot r\] called the action morphism, and it makes the orbit mapping \(\omega: \phi \to E; r \mapsto z \cdot r\) a topological embedding \(\forall z \in E\) [4].

\(\pi\) is an open surjective continuous mapping whose fibers \((\cdot):E \pi x\) are the orbits of \(z \in E\) under \(\Gamma\) with \(\pi(z)\).
(I.2.2.i) Proposition: Let $\phi : E \times \Gamma \to E$ be a principal action of a topological group $\Gamma$ on a topological space $E$, then the set of all orbits $G / E \times E / \Gamma$ is topological transitive groupoid of base $E \times E / \Gamma$ with the identification topology associated to the canonical mapping

$$\pi^\top : E \times E \to G, \pi^\top z', z [ \{ z', z \} ] [3] [4] [7].$$

*The groupoid $G / E \times E / \Gamma$ of a base $E \times E / \Gamma$ is called Ehresmann groupoid.

*Notice that whenever $\phi : E \times \Gamma \to E$ is a principal action of a top. group $\Gamma$ on a top. space $E$, then $\Gamma$ acts principally on $E \times E$ by the action defined by $\phi \cdot ((z', z), r) = (\phi (z', r), \phi (z, r)) \forall ((z', z), r) \in (E \times E) \times \Gamma$. The pair $(\pi^\top, \pi) : (E \times E, E) \to (G / E \times E / \Gamma, B / E / \Gamma)$ is a surjective open morphism in $TG$ [4].

*Let $f : X \to Y$ be a continuous mapping. Then $f$ is a (topological) submersion if $\forall x \in X$ there is an open neighborhood $U$ of $f(x)$ in $Y$ and a continuous right inverse $\sigma : U \to X$ of $f$ such that $\sigma (f(x)) = x$ [7].

* A topological group $\Gamma$ is said to act regularly on a topological space $E$ if it acts principally on $E$ and the canonical mapping $\pi : E \to B / E / \Gamma$ is a (topological) submersion [4].

(I.2.3) A continuous action $:\Psi : G \times (E \times \Gamma) \to E$ of a topological groupoid $(G, B)$ on a topological bundle $(E, P, B)$ is principal if its free and transitive action and for each $z \in E$ the orbit mapping $\iota : z \in G E \Psi \to$ is an open. Notice from the def. of principal action of a topological groupoid $G$, we have that each $\alpha$-fiber $z \in G$ is homeomorphic to $E$ [4].

(I.2.3.i) Proposition: Let $\phi : E \times \Gamma \to E$ be a law of principal (right) action of a topological group $\Gamma$ on a topological space $E$ then the Ehresmann groupoid $(G / E \times E / \Gamma, B / E / \Gamma)$ acts principally on $E$ by the action

$\Psi : G * E \to E$ defined by $\Psi ([z'', z'], z) = \phi (z'', T(z', z))$ where $T E \times E \to \Gamma$.
is the action morphism [4].

(1.2.4) [4][7] A locally trivial (L.T) groupoid is a topological groupoid 
\((G, B)\) such that \(\forall x \in B\) there is an open neighborhood \(U\) of \(x\) in \(B\) such 
that \(\tau: U \times T - U \times U\) \((G, B)\) is isomorphic to the trivial groupoid \(\times U\times G\times \times U\) in 
\(TG\).

\(G*E\) is the fiber product of \(a\) and \(P\) over \(B\).

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(1.2.4.i) Proposition: Let \(\Psi: G*E \to E\) be a law of a principal action of a 
L.T trivial groupoid \((G, B)\) on a topological space \(E\) then \(\Gamma\) acts 
regularly on \(E\) for a representative element \(\Gamma\) of the isomorphy class 
\((G, B)\) \([4]\).

II) Vector groupoid

(1) Algebraic case

(II.1) A vector groupoid is a groupoid \((G, B)\) with a property that each 
subset \(y \times G\) of \(G\) has a structure of vector space. Notice that every 
vector groupoid is a transitive groupoid.

* Let \(V\) be any vector space and \(B\) a (non-empty) set then:

(i) \(G \times B \times V \times B\) is a vector groupoid of a base \(B\) (trivial vector 
groupoid).

(ii) In (i); if \(B = pt\), we find that every vector space \(V\) is a vector 
groupoid of a singleton point base.

*A morphism of vector groupoids is a morphism of groupoids 
\((f, f) : (G, B)\) \((G, B)\) 
\((G', B')\) such that \(\rho : f/G \times y \times x \times y\) 
\(\tau\) is a linear mapping \(\forall x, y \in B\).

Notice that every linear mapping between any vector spaces is a 
morphism of vector groupoids.

(2) Topological case

(II.2.1) A topological vector groupoid is a vector groupoid \((G, B)\) 
(along with topologies on \(G\) and \(B\) such that \((G, B)\) becomes 
topological groupoid and each subset \(y \times G\) (with a subspace topology of 
\(G\) is a topological vector space.)
A morphism of topological vector groupoids is a morphism of vector groupoids \((f,f):(G,B) \to (G',B')\) such that \(f\) is continuous. We denote by \(TVG\) the category of topological vector groupoids and their morphisms; also, we denote by \(TV\) the category of topological vector spaces and Vector groupoid and Isometries.

An isomorphism of topological vector groupoids is a morphism of topological vector groupoids \((f,f):(G,B) \to (G',B')\) such that \(f\) is homeomorphism.

In the next proposition, we construct a structure of topological vector groupoid on Ehresmann groupoid \(G \times E / \Gamma\) that associates to an action of a special type of topological groups (that is the additive group \(\Gamma\) of a topological vector space \(V\), where \(\Gamma\) is a topological group) on a topological space \(E\). In the rest of our work, we shall use the symbol \(\Gamma\) to represent the additive group of a topological vector space \(V\) defined over a field \(K(R\) or \(C\)).

(II.2.2) Proposition: Let \(\phi: E \times \Gamma \to E\) be a law of principal action of a topological (additive) group \(\Gamma\) on a topological space \(E\) then the associated Ehresmann groupoid \((G \times E / \Gamma, B / \Gamma)\) is a topological vector groupoid.

Proof:

By proposition (I.2.2.i), the Ehresmann groupoid \(G \times E / \Gamma\) is a topological transitive groupoid. Let \(x \pi (z), y \pi (z') \in B / \Gamma\) (for \(z, z' \in E\)), each subset \(G \times E / \Gamma\) has the structure of a vector space over a field \(K(R\) or \(C\)) defined by:

(i) An addition law \(\gamma\) of \(G\) defined by:

\[
\gamma = \left(\begin{array}{c}
\gamma (x, y)
\end{array}\right) = \left(\begin{array}{c}
\gamma (x, y)
\end{array}\right)
\]

(ii) An exterior multiplication law \(\eta\) of \(G\) defined by:

\[
\eta = \left(\begin{array}{c}
\eta (x, y)
\end{array}\right) = \left(\begin{array}{c}
\eta (x, y)
\end{array}\right)
\]

is the addition law of \(\Gamma\).

(ii) An exterior multiplication law \(\eta\) of \(G\) defined by:

\[
\eta = \left(\begin{array}{c}
\eta (x, y)
\end{array}\right) = \left(\begin{array}{c}
\eta (x, y)
\end{array}\right)
\]
η′′′ η′ η ∀ a ∈ K and g [z′, r, z, r] s, t ∈ G,

where η
is the exterior multiplication law of Γ.

As a result From proposition (1.2.2.i), we get the following
commutative diagrams in T ;

A morphism of topological vector spaces is a linear mapping f : V → V′ which is continuous.

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Γ × Γ →
×.

(z, z) + η x E x E x G
× E −π−−→
γ, E x G

where ,

K × Γ →
×.

(z, z) + η x E x G
−π−−→
γ, E x G
K × G

in which (z, z) + η x E x G
π−−→
γ, E x G

are identification mappings (continuous open

surjective mappings ) and each of γ

η, η

is continuous. So γ’ and η’

are continuous. Hence γ, G is a topological vector space ∀ x, y ∈ B. Thus
the Ehresmann groupoid $G = E \times E / \Gamma$ is a topological vector groupoid of a base $E / \Gamma$.

* Let $(E, P, B)$ be a fiber bundle (with a property that each fiber $x E$ of $P$ is locally compact, locally connected, Hausdorff space), if we take $F(E)$ the set of all homeo morphisms from the fiber type $F$ onto the fibers of $P$, and $I(E)$ the set of all homeo morphisms between the fibers of $P$ (that induced from the element of $F(E)$). $F(E)$ & $I(E)$ are L.T transitive groupoids of a base $B$ [3] [7].

(II.2.3) A locally trivial vector (L.T.V) groupoid is a topological vector groupoid $(G, B)$ such that for each $x \in B$ there is an open neighborhood $U$ of $x$ in $B$ such that $\tau^{-1}(U \times U)$ is isomorphic to the trivial vector groupoid $x \times U \times G \times U$ in TVG.

* Let $(E, P, B)$ be a topological vector bundle (with a property that each fiber $x E$ of $P$ is locally compact, locally connected, Hausdorff space),

For more details about a fiber bundle & a topological vector bundle see [7].

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the sets $F(E), I(E)$ are L.T.V groupoids of a base $B$ (their elements are isomorphisms in TV).

(II.2.3.i) Proposition: Let $(G, B)$ be a locally trivial vector groupoid. If the right translation mappings between the fibers of $x \alpha$ are linear $\forall y \in B$ then $(x, , ,) \beta B$ is a topological vector bundle $\forall x \in B$.

Proof:
Let $x \in B$, the $x \beta$ fibers defined by:
$y G y B^{x \alpha \beta} \forall \in \beta ()$;
from the hypothesis $(G, B)$ is a topological transitive groupoid, then we have a homeomorphy class of topological spaces $x x \beta \alpha G$, and we shall take $F$ as a representative element of this class to be the fiber type of $x G$. By definition (II.2.1), each fiber $x \alpha G$ of $\beta$ and $F$ are topological vector spaces.

Now from the local trivial property of a topological vector
groupoid \((G,B)\) (def. II.2.3) and for \(y \in B\) we have:

\(\exists U\) open neighborhood of \(y\) in \(B\) such that \(U \times F \times U\) is isomorphic to

\(\cdot ( )\) \(v \cdot U \times U \quad G\) in \(TVG\), hence there is a homeomorphism

\(: v \cdot f \ U \times F \times U \to G\) with a commutative diagram in \(S\) (the category of sets and mappings);

\(v \cdot U \ F \ U \ f \ G \times \times \longrightarrow \)

\(pr\)

\(pr \beta \ ' \ a \ ' \) ............................... (1)

\(U\)

\(I\)

\(U \to \to \to \to \)

We have two cases about \(x \in B\); either \(x \in U\), then we take \(\psi\) to

be the restriction of \(f\) to

\(U \ F \ U \ F \times \times \ G \cdot ( U ) \ v \times x\)

\(- \times \to \times \to \to \beta\);

or \(x \notin U\), then we define \(\psi\ '\) to be the restriction of \(f\) to

\(y \times F \to U \times F \times y \to G\); but \(G\) is transitive groupoid, hence:

\(y \times \exists g \in G\) and a homeomorphism \(g \cdot U \times U \ R : G \to G\) defined by \(R h h g \cdot ( )\)

\(v \ y \ \forall h \in G\);

and in this case we take \(g \cdot U \psi\quad R \circ \psi\ ' : U \times F \to G\).

In both cases, we get that \(v \cdot x \psi\ U \times F \to G\) is a homeomorphism and satisfying:

\(y \cdot g \cdot y \cdot \beta ( \psi \ ( \cdot ) )\quad \forall y \ ' \in U, g \in F\) (by (1) and the definition of \(\psi\));

from the hypothesis \(f\) is an isomorphism in \(TVG\) (and the restriction

of \(\cdot R\) to \(y \times y \times G\) \(\to\) is linear \(\forall y \ ' \in U\)), then the restriction of \(\psi\) to

\(F \ y \ F G \cdot ( y ) \times x \to \ ' \times \to \ ' \cdot \beta\)

\(\cdot \beta\) is an isomorphism in \(TV\) \(\forall y \ ' \in U\). Hence

\((G, , B) \times \times \beta\) is a topological vector bundle \(\forall x \in B\).

(II.2.4) Proposition: Let \(\phi : E \times \Gamma \to E\) be a law of regular action of a

topological (additive) group \(\Gamma\) on a topological space \(E\) then
$(E, \pi, B) \xrightarrow{E/\Gamma}$ is a topological vector bundle.

Proof:

If we take $F \xrightarrow{\Gamma}$ as a fiber type of $E$ then each fiber $E$ of $\pi$ and $F$ are topological vector spaces (see the def. of principal action of top. group). Let $x \xrightarrow{\pi} (z) \in B$ (for $z \in E$), from the hypothesis $\phi$ is a regular action then:

$\exists$ open neighborhood of $x$ in $B$ and a continuous right inverse $\sigma : U \rightarrow E$ of $\pi$ such that $\sigma (\pi (z)) = z$.

Define $\nu \psi U \times F \rightarrow E$ by the following composition of mappings:

\[
\begin{array}{ccc}
U & \xrightarrow{F} & F \\
\times & \sigma & \\
\end{array}
\]

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$(y, r) \xrightarrow{\sigma ' (y), r} \phi ' (\sigma ' (y), r)$, where $\sigma '$ is the mapping that induced from $\sigma$ which maps $U$ onto its image $\nu E$ (under $\sigma$) and $\nu \phi \phi EF \times ' \xrightarrow{}$, so $\psi$ is continuous surjection and satisfying:

$\pi (\psi (y, r)) \ x \forall y \in U \text{ and } r \in F$ ............................................... (2)

$\psi$ is an injective mapping, since:

for each $(y, r), (y', r') \in U \times F$, if

\[
\begin{align*}
\psi (y, r) \ x \psi (y', r') \Rightarrow y \ x \pi (\psi (y, r)) \ x \pi (\psi (y', r')) \ x \ y' \ (\text{by (2)}
\end{align*}
\]

$\Rightarrow \phi (\sigma ' (y), r) \ x \phi (\sigma ' (y), r') \Rightarrow r \ x r' \ (\phi \text{ is free action}) \Rightarrow (y, r) \ x (y', r')$;

hence from the def. of $\psi$, we find that $\nu \psi : U \times F \rightarrow E$ is a homeomorphism such that the restriction of $\psi$ to $F \rightarrow E$ is an isomorphism in $TV \forall y \in U$. Therefore $(E, \pi, B) \xrightarrow{E/\Gamma}$ is a topological vector bundle.

* As a result from the preceding proposition, we get that $\forall z \in E$ and $x \xrightarrow{\pi} (z) \in B$, $: x \phi \Gamma \rightarrow E$ is an isomorphism in $TV$.

(II.2.5) Proposition: Let $(G, B)$ be any topological groupoid acts
principally on a topological vector bundle \((E, P, B)\) then \((G, B)\) is a topological vector groupoid.

Proof:
From the hypothesis, we can get that \(G\) is a topological transitive groupoid and each subset \(y \times G\) has a structure of topological vector space (with a subspace topology of \(G\)) (see def. l.2.3). Hence \((G, B)\) is a topological vector groupoid.

* From proposition (ll.2.5), we have that \(\forall z \in E \text{ and } x, y \in B\) (with \(x \in P(z)\), :
\[
z, G \times y, y \times G \rightarrow \ast \\
\Psi \rightarrow \text{ is an isomorphism in } TV.
\]

(ill) Regular action of topological groupoid and topological group

(lll.1) Let \((G, B)\) be any topological groupoid and \((E, P, B)\) a topological vector bundle, then \(G\) is said to act linearly on \(E\) if it acts continuously on \(E\) and the mapping \(\psi \rightarrow \text{ is a vector space isomorphism} \forall g \in G [7].\)

(lll.2) A topological groupoid \((G, B)\) is said to act regularly on a topological vector bundle \((E, P, B)\) if it acts principally and linearly on \(E\).

(lll.2.1) Proposition: Let \(\psi \rightarrow \text{ be a law of a regular action of a topological groupoid } (G, B)\) on a topological vector bundle \((E, P, B)\) then \((G, B)\) is locally trivial vector groupoid.

Proof:
With the given in the hypothesis and by using prop. (ll.2.5), we get that \((G, B)\) is topological vector groupoid and for \(x \in P(z) \in B\) (with \(z \in E\)) we have:

\(\exists U \text{ open neighborhood of } x \text{ in } B \) and a homeomorphism \(\psi \rightarrow U \times F \rightarrow E\) such that:
\[
P(\psi(y, a)) \quad \forall y \in U \text{ and } a \in F \text{ (the fiber type of } E) \quad \ldots \ldots \quad (1)
\]
Let \(\psi \rightarrow \sigma \rightarrow U \rightarrow G\) be the mapping defined by the following composition:
$u, y, a, x, F, E, G$ $F \rightarrow z \rightarrow G$

$\Psi \rightarrow \{ \} \ldots (2)$

$y, a, y, a, g \circ o(,)(,) \psi(,)$, where $(,)(,)^{\psi} g \psi y a o$;

for a fixed element $"o a$ in $F$, and $z \in F E$

$\psi, (,)$

$\Psi$ is the orbit

mapping at $z$) defined by $F z g :(,)^{(')}$, where $v z g z E \Psi(,) ^{(')} \epsilon$. From def.

(l.2.3) and the remark after prop. (lI.2.5), we find that $z F$ is a

homeomorphism and its restriction to $y x E \rightarrow G$ is an isomorphism in $T V$

$\forall y \in U$ ........................................... (3)

From (1), (2), $\sigma$ is continuous – right inverse of $: x, \beta \rightarrow U$; hence

$U G U x, x \times$ is isomorphic to $v v G$ in $T G$ (see [7]) by a homeomorphism

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$f: U \times x, G, x U v v \rightarrow G$ defined by the following composition:

$\mu \sigma \sigma \mu$

...(4)

where $\sigma \prime \prime o \sigma ($
the restriction of the inversion law \( v \) of \( G \) on \( U \times G \) and

\[
U \times \mu' \quad \mu G \times G ,
\]

\[
U \times \mu'' \quad \mu G \times G .
\]

Now since \( \Psi \) is a linear action (def. III.1) and from (3), we have that for each \( g \in G \) (let \( \alpha(g) \), \( \beta(g) \), \( y, y' \in U \)) the following diagram is commutative in \( TV \):

\[
\begin{array}{cccccccc}
F & G & g & G & \times & F & E & E
\end{array}
\]

\[
\begin{array}{cccccccc}
\Rightarrow & \Psi & \times & \Rightarrow & \mu & \mu ;
\end{array}
\]

in which each of \( \mu \), \( \Psi \), and \( \mu \) are isomorphisms in \( TV \), so \( \mu \) is an isomorphism in \( TV \) .............................. (5)

By (4), (5), the restriction of \( f \) to \( U \times G \) \( \times \times \) is an isomorphism in \( TV \forall y, y' \in U \), hence \( U G U \times \times \times \) is isomorphic to \( U G \).
in $TVG$. Thus $(G,B)$ is a locally trivial vector groupoid.

*From the preceding proof of prop. (Ill.2.1), we get that whenever $\Psi$ is a principal action of a topological groupoid $(G,B)$ on a fiber bundle $(E,P,B)$ then $G$ is L.T groupoid.

*From the remark after prop. (II.2.4), we get that for $g \in G$ (with $\alpha(g) P(z) x, \beta(g) P(z') y$) the following diagram is commutative in $TV$

\[ \begin{array}{c}
\begin{array}{ccc}
F & G & E \\
\downarrow & \downarrow & \downarrow \\
\Psi & \Psi & \Psi \\
\end{array}
\end{array} \]

\[ \begin{array}{ccc}
G & F & E \\
R_g & \Psi & \Psi \\
\end{array} \]

in which $\Psi$ are isomorphisms in $TV$. So, $R_g$ is an isomorphism in $TV$.

(Ill.2.2) Corollary: Let $(G,B)$ be any topological groupoid acts regularly on a topological vector bundle $(E,P,B)$ then $(\cdot, \cdot)_{G,B}$ is a topological vector bundle $\forall x \in B$.

Proof: Using propositions (Ill.2.1), (II.2.3.i) and the above remark.
(III.2.3) Proposition: Let \( G E E \Psi \ast \rightarrow \) be a law of a regular action of a topological groupoid \((G,B)\) on a topological vector bundle \((E,P,B)\) then \((, , )_x G \beta B \) and \((E,P,B)\) are isomorphic in \( TVB \ \forall x \in B \).

Proof:
Let \( x \in P(z) \in B \) (for \( z \in E \)), from def. (I.2.3) and the remark after prop. (II.2.5), \( \zeta : G \rightarrow E \) is a homeomorphism and its restriction to \( y_x, G \rightarrow E \) is an isomorphism in \( TV \ \forall y \in B \), and the square:

\[
\begin{array}{ccc}
G & \xrightarrow{\psi} & E \\
\downarrow & & \downarrow \\
\beta & & \\
p & \mapsto & B \\
\end{array}
\]

is commutative in \( S \).

Hence \((, , )_x G \beta B \rightarrow E P B \) is isomorphism in \( TVB \ \forall z \in E \) and \( x \in P(z) \in B \).

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(III.2.4) Proposition: Given \((G,B)\) is a topological groupoid acts regularly on a topological vector bundle \((E,P,B)\) by the action \( \Psi : G \ast E \rightarrow E \) then \( G \) acts regularly on \((, , )_x G \beta B \ \forall x \in B \).

Proof:
For \( x \in P(z) \in B \) and \( z \in E \), the following diagram

\[
\begin{array}{ccc}
& \ast & \rightarrow & \ast \\
& \times \Psi & & \\
& \mu & & \\
\end{array}
\]

is commutative in \( S \) ....... (1)
where \( \Psi \) is a continuous mapping, since it is the restriction of \( \mu \) on \( G \). Also, \( \Psi \) is an action of \( G \) on \( G \), since:

1. \( ( , ) ( , ) \) is \( \mu \) \( \beta \) \( \Psi \) \( \forall \) \( G \).
2. \( ( , ) ( , ) \) is \( \beta \) \( \Psi \) \( \mu \) \( \forall \) \( g g' \in G \).
3. \( \Psi(g, h) \) \( \mu(g, h) \) \( \Psi(g, g', h) \) \( \forall(g, g', h) \in G \).

Let \( , h \in G \) such that \( \Psi(g, g') \Rightarrow \mu(g, g') \) \( g' \) then \( g \ \omega(\alpha(g')) \), hence \( \Psi \) is a free action. For any \( , h \in G \);

\[ \Psi(h' h^{-1}, h) \Rightarrow \mu(h' h^{-1}, h) \Rightarrow h', \] so \( \Psi \) is a transitive action.

Now let \( , h \in G \) with \( , \beta h \) \( y \in B \), then from (1) and def. (1.2.3), the following diagram is commutative in \( T \);

For each \( g \in G \), \( g \) has unique right unity \( \omega(\alpha(g)) \) and unique left unity \( \omega(\beta(g)) \) [1] [7].

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\[ \psi', \psi \rightarrow . \]

\[ , \text{ where } ( ) : z' \Psi h \; ; \]

in which

\( \Psi \) and \( F' \) are homeomorphisms, hence \( \Psi \) is a
homeomorphism (open) \( \forall h \in G \). Thus \( \Psi' \) is principal action. Also, since \( \Psi \) is a linear action then by (1) and remark after prop. (\( \|2.5 \)) and for each \( g \in G \) the following diagram is commutative in \( TV \);

\[
\begin{array}{c}
\varepsilon \\
\vdash \\
\psi \\
\Psi \\
\end{array}
\]

\[
\begin{array}{c}
\alpha \\
\vdash \\
\beta \\
\end{array}
\]

\( \gamma \)

\( \delta \)

\( \varepsilon \)

\( \zeta \)

\( F \)

\( G \)

\( G F F E \)

\( \Psi' \)

\( \Psi \)

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groupoid using Ehresmann (vector) groupoid.
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(III.4) Proposition: Let \( \phi: E \times \Gamma \to E \) be a law of a regular action of a
topological (additive) group \( \Gamma \) on a topological space \( E \) then the
Ehresmann groupoid \( (G \ E \times \ E / \Gamma \ , \ B \ E / \Gamma) \) acts regularly on \( (E , \pi , B) \).
Proof:
From the hypothesis \( \phi \) is a principal action, then by prop. (I.2.3.i)
\( G \) acts principally on \( E \) by the action \( \Psi : G \times E \to E \) defined by:
\[
\Psi : ([z'', z'], z) \to (z'', T(z', z)) \quad \forall ([z'', z'], z) \in G \times E \quad \cdots \quad (1)
\]
where \( G \times E \) is the fiber product of \( \alpha \) and \( \pi \) (whose elements \((z'', z'), z\))
with \((z', z) \in E \times E \), and \( T \) is the action morphism associated with \( \phi \).
Since \( \phi \) is a regular action, then \((E , \pi , B) \) is a topological vector
bundle (prop. II.2.4). Let \( g \ [z'', z'] \in G \) with \( \alpha (g) \pi (z') x \)
, \( \beta (g) \pi (z'') y \in B \), the mapping \( g_{\times} \Psi : E \to E \) is defined by:
\[
g_{\times} \Psi : z \to T(z', z) \quad \forall z \in E \cdot
\]
\( \cdot \) \( \phi \) \( (\cdot) \), where
\[
\{ \cdot T(z', z) \times E' \}
\]
From the remark after prop. (II.2.4), we have the following
commutative diagram in \( TV \);
and \( z \phi \) are isomorphisms in \( TV \), so \( g \) is an isomorphism in \( TV \forall g \in G \). Thus \( \Psi \) is linear action, and therefore \( G \) acts regularly on \( (E,P,B) \).

IV) Topological vector groupoid and isometric action of topological groupoid

(IIV.1) Let \( (G,B) \) be any topological groupoid and \( (E,P,B) \) an \( n \)-dimensional topological vector bundle \(^1\), then \( G \) is said to act isometrically on \( E \) if it acts continuously on \( E \) and the mapping \( g \mapsto \Psi_g \) is an isometry \(^2\) \( \forall g \in G \).

1 For more details about an \( n \)-dim. topological vector bundle and an isometry see [5] [6] .

2 Each fiber \( E \) in \( n \)-dim. top. v. bundle has a structure of \( n \)-dim. v. space, hence it has a structure of normed space and then we can talking about an isometry between the fibers.

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* If \( (E,P,B) \) is an \( n \)-dimensional topological vector bundle. Take \( G \) IS \( E \) the set of all isometries between the fibers of \( P \), then \( G \) is a L.T.V. groupoid which acts isometrically on \( E \) by the action \( \Psi : G \ast E \rightarrow E \) defined by \( \Psi(f,z) = f(z) \forall (f,z) \in G \ast E \).

* Every isometric action of any topological groupoid on \( n \)-dim. topological vector bundle is a linear action.

Using a special kind of topological groups such as the underlying additive group \( \Gamma \) of an \( n \)-dimensional vector space \( V \) defined over a field \( K(R \) or \( C) \) (where \( \Gamma \) is a topological group with the topology that induced from a (one-to-one) correspondence between \( \Gamma \) and \( K \)). and a special kind of topological vector groupoids such as the Ehresmann groupoid \( (E \times E / \Gamma , E / \Gamma) \), we link the concept of regular action of a topological group with the concept of isometric action of a topological groupoid, as it follows in the next propositions.

(IIV.2) Proposition: Let \( \phi : E \times \Gamma \rightarrow E \) be a law of a regular action of a topological (additive) group \( \Gamma \) on a topological space \( E \) then \( (E,\Pi , B \ E / \Gamma) \) is \( n \)-dimensional topological vector bundle.

Proof :
In a similar way to the proof of prop. (III.2.4) and by taking $F \ K_n$ (which is homeomorphic to $\Gamma$ in $T$) as a fiber type of $E$.

(IV.3) Proposition: Let $\phi : E \times \Gamma \to E$ be a law of a regular action of a topological (additive) group $\Gamma$ on a topological space $E$. Then the associated Ehresmann groupoid $(G \ E \times E / \Gamma, B \ E / \Gamma)$ acts isometrically on $(E, \pi, B)$.

Proof: From the hypothesis $\phi$ is a regular action, then by propositions ((IV.2), (III.4)), $(E, \pi, B)$ is an $n$-dimensional topological vector bundle and the Ehresmann groupoid $G \ E \times E / \Gamma$ acts linearly on $E$ by the action $\Psi : G * E \to E$ defined by:

$\Psi([z'', z'], z) \phi(z'', T(z', z)) \forall ([z'', z'], z) \in G * E$.

For $g \ [z', z] \in G$ with $\alpha(g) \pi(z') x, \beta(g) \pi(z') y \in B$, we have the following commutative diagram in the category of normed spaces and isometries;

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∀g ∈ G. Thus Ψ is an isometric action of G on (E, π, B).

REFERENCES