Semi-injectivity and Continuity
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Abstract
In this paper we related semi-injectivity and quasi-continuity. Conditions are considered under which we describe semi-injectivity and continuity. Finally we studied semi-injective modules over generalized uniserial rings.

Introduction
An R-module M is quasi-injective, if each R-homomorphism of a submodule of M into M can be extended to an R-endomorphism of M [1]. V. Govorov in [2], introduced the concept of semi-injective modules as a generalization of quasi-injective modules. An R-module M is said to be semi-injective, if each R-endomorphism of a submodule of M can be extended to an R-endomorphism of M. Y. Utumi in [3] studied continuous rings as a generalization of selfinjective rings. Jeremy [4], Mohamed and Bouhy [5] and Goel and Jain [6] generalized these ideas to modules. The concept of psuedoinjective modules was introduced by Jain and singh in [7] as a generalization of quasi-injective modules. An R-module M is pseudoinjective if each R-monomorphism of a sub module of M into M can be extended to an R-endomorphism of M. H. Q. Dinh in [8] showed that every pseudo-injective extending module is continuous. In this work, we show that every semi-injective module is quasi-continuous. Recall that, a submodule N of M is closed, if N has no proper essential extension in M. several properties of closed submodules are studied in semi-injective modules. As a consequence of these properties, we show that the class of semi-injective modules is contained in that of some generalization of quasi-injective modules. A submodule N of an R-module M is stable, if \( \alpha(N) \subseteq N \) for each \( \alpha \in \text{Hom}_R(N,M) \), and M is called fully stable, in case each submodule of M is stable [9]. This is equivalent to saying that
each cyclic submodule of \( M \) is stable. It is also known that, \( M \) is a fully stable \( R \)-module if, and only if, \( \text{ann}_M(\text{ann}_R(x)) = (x) \) for each \( x \in M \) [9]. Semi-injective modules were generalized to that of Cl-semiinjective in [10]. Also we introduced some generalization of fully stable modules, namely, generalized fully stable modules. These concepts are used to established that every semi-injective generalized fully stable module is continuous. We enclosed this paper by studying semi-injective modules over generalized uniserial rings. We prove that every semi-injective module over a generalized uniserial ring is quasi-injective. Finally, we remark that all rings considered in this work are commutative with identity unless otherwise stated, and all modules are left unitary.

§ 1. SEMI-INJECTIVE MODULES AND CONTINUOUS MODULES

Let \( M \) be an \( R \)-module, \( E = E(M) \) be its injective envelope and \( S = \text{End}(E) \) be the ring of endomorphisms of \( E \). In [1], R.E.Johnson and E.T.Wong proved that \( M \) is quasi-injective if and only if, \( SM \subseteq M \). V. Govorov introduced the concept of semi-injective modules as a generalization of quasi-injective modules. An \( R \)-module \( M \) is said to be semi-injective, if for every submodules \( N \) of \( M \), each \( R \)-endomorphism of \( N \) extends to an \( R \)-endomorphism of \( M \) [2]. An \( R \)-endomorphism \( \alpha \in S \) is called essential endomorphism if \( \alpha(N) \subseteq N \) for some essential submodule \( N \) of \( M \). Let \( K_e \) be the set of all essential endomorphisms in \( S \). The following theorem appeared in [11].

Theorem (1.1): The following statements are equivalent for an \( R \)-module \( M \).

(a) \( M \) is semi-injective and \( J(S)M \subseteq M \).
(b) \( K_e M \subseteq M \), where \( J(S) \) is the Jacobson radical of \( S \).

Theorem (1.2): Let \( M \) be a semi-injective \( R \)-module. Then \( M \) is invariant under each idempotent in \( S \).

Proof: For each \( f = \text{id} \in S \), if \( x \in f(M \cap f^{-1}(M)) = f(M) \cap M \), then \( x = f(m) \in M \) for some \( m \in M \). \( f(x) = f(f(m)) = f(m) = x \), hence \( x \in M \cap f^{-1}(M) \). Therefore \( f : M \cap f^{-1}(M) \rightarrow M \cap f^{-1}(M) \) Semi-

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injectivity of \( M \) induces an \( R \)-endomorphism \( h : M \rightarrow M \).

Since \( E \) is injective, there is \( g \in S \) such that \( g(M) = h(M) \subseteq M \).

Also \( (g - f)(M \cap f^{-1}(M)) = 0 \). Since \( g(M) \subseteq M \), we have \( M \cap (g - f)(M) \subseteq M \cap f^{-1}(M) \subseteq \text{Ker}(g - f) \), thus \( (g - f)(M) \cap M = 0 \).

But \( M \) is an essential submodule of \( E \), then \( (g - f)(M) = 0 \) and hence \( f(M) = g(M) \subseteq M \).

Corollary (1.3): Let \( M \) be a semi-injective \( R \)-module and \( E \) be its injective envelope. If \( \alpha \in S \),
\[ \Lambda \in \oplus, \text{ then } M (M \in E) \alpha \alpha = \oplus \cap \in \Lambda. \]

**Proof:** For each \( \alpha \in \Lambda \), let \( \pi_\alpha : E \rightarrow E_\alpha \) be the natural projection of \( E \) onto \( E_\alpha \). Theorem (1.2) implies that \( \pi_\alpha (M) \subseteq M \) for each \( \alpha \in \Lambda \), hence \( M \pi (M) (M E_\alpha ) \subseteq M. \)

Y. Utumi in [3] introduced continuous ring as a generalization of self-injective rings. These concepts of continuity and quasi-continuity were extended to modules by L. Jeremy [4], S. Mohamed and T. Bouhy [5], V. Goel and S. K. Jain [6]. The notion of quasi-continuous modules which effectively extends that of continuous modules seems now more fundamental. An R-module \( M \) is said to be continuous if (a) every closed submodule of \( M \) is a direct summand of \( M \) and (b) every submodule of \( M \) which is isomorphic to a direct summand of \( M \) is a direct summand of \( M \). An R-module \( M \) is called quasi-continuous, if \( M \) has condition (a) and condition (c): for any direct summands \( P, N \) of \( M \) with \( P \cap N = 0 \), \( P \oplus N \) is also a direct summand of \( M \). Its well-known that, injectivity \( \rightarrow \) quasi-injectivity \( \rightarrow \) continuity \( \rightarrow \) quasi-continuity.

For semi-injective, we have the following theorem (1.4): Every semi-injective module is quasicontinuous.

**Proof:** Let \( M \) be a semi-injective R-module. To prove condition (a), it is enough to show that, every submodule of \( M \) is essential in a direct summand of \( M \)[12]. If \( N \) is any submodule of \( M \), then \( E(M) = E(N) \oplus K \) for some submodule \( K \) of \( E(M) \). Corollary (1.3) implies that \( M = (M \cap E(N)) \oplus (M \cap K) \), \( N \) is an essential submodule of \( M \cap E(N) \). For condition (c), let \( K \) and \( L \) be direct summands of \( M \) with \( K \cap L = 0 \). Then \( E(N) = E(K \oplus L) \oplus E = \oplus E(K) \oplus E(L) \oplus E \) for Semi-injectivity and Continuity.

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some submodule \( E \) of \( E(M) \). Since \( K \) and \( L \) are direct summands and essential in \( M \cap E(M) \) and \( M \cap E(L) \) respectively. This completes the proof.

**Proposition (1.5):** Let \( M \) be a semi-injective R-module. If \( N_1 \) and \( N_2 \) are submodules of \( M \) with \( N_1 \cap N_2 = 0 \), then there exist
submodules
$M_1, M_2$ of $M$ such that $M=M_1 \oplus M_2$ and $N_i \subseteq M_i$ ($i=1,2$).
Proof: By theorem(1.4), there exist submodules $K_1$ and $K_2$ of $M$ such that $N_i$ is essential in $K_i$ ($i=1,2$). Clearly, $K_1 \cap K_2 = 0$.
Again theorem(1.4) implies that $K_1 \oplus K_2$ is a direct summand of $M$, hence $M = K_1 \oplus K_2 \oplus K$ for some submodule $K$ of $M$. Note that, $N_i \subseteq K_1$ and $N_i \subseteq K_2 \oplus K$.

Lemma(1.6): Let $M_1$ and $M_2$ be $R$-modules and $M = M_1 \oplus M_2$. If $L_i$ is a closed submodule of $M_i$ ($i=1,2$), then $L_1 \oplus L_2$ is a closed submodule of $M$.
Proof: Let $K$ be an essential extension of $L_1 \oplus M_2$ in $M$. Then $M_2 \subseteq K$ implies that $K = K \cap M = (K \cap M_1) \oplus M_2$. Now $L_1 = (L_1 \oplus M_2) \cap M_1$ is essential in $K \cap M_1$, hence $L_1 = K \cap M_1$ and $L_1 \oplus M_2 = K$. Thus $L_1 \oplus M_2$ is closed in $M$. Similarly, $L_1 \oplus L_2$ is closed in $L_1 \oplus M_2$, thus $L_1 \oplus L_2$ is closed in $M$.
The proof of the following corollary follows from lemma (1.6) and theorem (1.4)

Corollary (1.7): Let $M$ be a semi-injective $R$-module. If $N_1$ and $N_2$ are closed submodules of $M$ with $N_1 \cap N_2 = 0$, then $N_1 \oplus N_2$ is a closed submodule of $M$.
The following corollary shows, that the class of semiinjective modules is contained in that of some generalization of quasi-injective modules.

Corollary (1.8): Let $M$ be a semi-injective $R$-module. If $N_1$ and $N_2$ are closed submodules of $M$ with $N_1 \cap N_2 = 0$, then each $R$-homomorphism $f: N_1 \oplus N_2 \to M$ can be extended to an $R$-endomorphism of $M$.

Recall that, an $R$-module $M$ is said to be quotient essential noetherian (Simply QEN), if each ascending chain $L_1 \subseteq L_2 \subseteq \ldots$ of $M$ with $L_{i+1}/L_i$ is essential in $M/L_i$ for each $i$, there is a positive integer $n_0$ such that $L_n = L_{n+1}$ for all $n \geq n_0$. A Al- Mustansiriya J. Sci Vol. 18, No 1, 2007 103

ring $R$ is QEN ring if it is QEN $R$-module. The following was proved in [11]: if $R$ is a QEN ring, then $M$ is a semi-injective $R$-module if and only if, $M$ is invariant over $K_e$.

We note that, semi-injectivity is not closed under submodules, see example(1.13). But in theorem(1.4), we show that closed submodules, inherit semi-injectivity. In this regard we consider certain conditions under which each proper cyclic submodule of a semi-injective module is semi-injective.

First, we need the following lemmas.
Lemma(1.9): Let $M$ be an $R$-module and $x \in M$. If $E$ is $R$-injective
envelope of $\mathbb{R}/\text{ann}^{\mathbb{R}}(x)$, then $E_1=\text{ann}^{\mathbb{E}}(\text{ann}^{\mathbb{R}}(x))$ is the $\mathbb{R}/\text{ann}^{\mathbb{R}}(x)$-injective envelope of $\mathbb{R}/\text{ann}^{\mathbb{R}}(x)$.

Proof: write $H=\text{ann}^{\mathbb{R}}(x)$. To show that $E_1$ (as $R/H$-module) is an essential extension of $R/H$, if $y$ is a non-zero element in $E_1$, then $y\in E$ and $Hy=0$. Since $E$ is an essential extension of $R/H$ (as $R$-module), then there is $r(\neq 0)$ in $R$ such that $ry(\neq 0)\in R/H$. Thus $Hr\in R/H$ and $(Hr)y \neq H$, hence $E_1$ is an essential extension of $R/H$. For each ideal $I$ of $R/H$ and $R/H$-homomorphism $\alpha : I \to E$

- Define $\alpha : I \to E$
- (where $I=I/H$)
- by $(i)(iH)$
- $\alpha = \alpha$ where $(\ ) = 0$
- $\alpha iH H$ for each $i \in I$. It is a matter of checking that $\alpha$ is well-defined $R$-homomorphism. Injectivity of $E$ implies that there is $e \in E$ such that for each $w \in I$, $\alpha (w)=we$. Now $He=Hwe$.

Lemma (1.10): Let $M$ be an $R$-module and $I$ be an ideal of $R$. If $M$ is a semi-injective $R/I$-module, then $M$ is a semi-injective $R$-module. Conversely, if $M$ is a semi-injective $R$-module such that $I \subseteq \text{ann}(M)$, then $M$ is a semi-injective $R/I$-module.

Proof: The relation $(r+I)m=rm$ for each $r \in R$ and $m \in M$ is used in each case to define $M$ as a module over $R$ (or $R/I$) where is given as a module over $R/I$ (or $R$). It is then easy to see that the concepts of submodules and endomorphisms coincide over each ring.

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Theorem (1.11): Let $M$ be a uniform module over $\mathbb{QEN}$ ring $R$. Then the following statements are equivalent
(1) $M$ is a fully stable $R$-module.
(2) $R/\text{ann}^{\mathbb{R}}(x)$ is self semi-injective ring for each $x \in M$. 
Each proper cyclic submodule of $M$ is semi-injective.

Proof: (1) → (2): Since $M$ is uniform, then for each $x \in M$, the cyclic submodule $(x)$ is essential in $M$. Hence $E(R/\text{ann}_R(x)) \cong E(x) = E(M) = E$. By lemma (1.9), $E = \hat{\text{ann}}_E(\text{ann}_R(x))$ is the injective envelope of $R/\text{ann}_R(x)$ as $R/\text{ann}_R(x)$-module. Since $M$ is fully stable, then $(x) = \text{ann}_M(\text{ann}_R(x)) \supseteq \text{ann}_E(\text{ann}_R(x))$, thus $E = \hat{\text{ann}}_E(\text{ann}_R(x)) = (x) = R/\text{ann}_R(x)$, so $R/\text{ann}_R(x)$ is the injective envelope of $R/\text{ann}_R(x)$ as $R/\text{ann}_R(x)$-module. Therefore $R/\text{ann}_R(x)$ is self-injective (and hence self-semi-injective) ring.

(2) → (3): For each $x \in M$, if $L$ is a submodule of $(x)$ and $f: L \to L$ is an $R$-endomorphism, then $\text{ann}_R(x) \subseteq \text{ann}_R(L)$ that is, $L$ is annihilated by $\text{ann}_R(x)$. Lemma (1.10) implies that $L$ is an $R/\text{ann}_R(x)$-module. There exists an $R$-endomorphism $g$ of $R/\text{ann}_R(x)$ which extends $f$ to $(x)$. Thus $(x)$ is a semi-injective $R$-module.

(3) → (1) Let $(x)$ be any cyclic submodule of $M$ and $\beta: (x) \to M$ be an $R$-homomorphism. Since $M$ is uniform, then $E(x) = E(M)$. Injectivity of $E(M)$ implies that, there exists $\gamma: E(x) \to E((x))$. But $E(M)$ is uniform, then $\ker(\gamma)$ is essential in $E(M)$, thus $\gamma \in J(\text{End}_R(E(M)))$. Since $J(\text{End}_R(E(M))) \subseteq K_\epsilon$, thus $\gamma \in K_\epsilon$. By (3), $(x)$ is semi-injective $R$-module, then $\gamma((x)) \subseteq (x)$ and hence $\beta((x)) \subseteq (x)$. Therefore $M$ is fully stable.

Corollary (1.12): Every uniform fully stable module over noetherian ring is semi-injective.

Example (1.13): The following lemma was proved in [13]: Let $M$ be an $R$-module whose lattice of submodules is $\omega$. Then $M$ is a fully stable $R$-module but not semi-injective [11]. R. Hallett in [14] gives an example of module which satisfies the conditions of the above lemma. This example shows that the uniform property of the module in corollary (1.2) is essential. Further, $E(M)$ is semi-injective and $M$ as a submodule of $E(M)$ is not semi-injective. The converse of corollary (1.12) is not true in general, for example, the ring $\mathbb{Z}$ of integers is uniform semi-injective but not fully stable [9].

In this part we investigate the following concept to clarify the relation between semi-injective module and continuous modules.

First, we recall that an $R$-module $M$ is called $CI$-fully stable, if for each closed submodule $C$ of $M$, $\alpha(C) \subseteq C$ for each $R$-homomorphism $\alpha: M \to M$. Where $N_1$ is not isomorphic to $N_2$, then $M$ is a fully stable $R$-module [9] but not semi-injective [11]. R. Hallett in [14] gives an example of module which satisfies the conditions of the above lemma. This example shows that the uniform property of the module in corollary (1.2) is essential. Further, $E(M)$ is semi-injective and $M$ as a submodule of $E(M)$ is not semi-injective. The converse of corollary (1.12) is not true in general, for example, the ring $\mathbb{Z}$ of integers is uniform semi-injective but not fully stable [9].
Definition (1.14): An R-module M is called a generalized fully stable if for each closed submodule N of M which is isomorphic to a closed submodule of M, \( \alpha(N) \subseteq N \) for each \( \alpha \in \text{Hom}_R(N,M) \).

It is clear that every generalized fully stable module is Cl-fully stable.

Lemma (1.15): Let M be a generalized fully stable R-module and \( C_1, C_2 \) be two submodules of M with \( C_1 \) is a closed in M. If \( C_1 \neq C_2 \), then \( C_1 \) is not isomorphic to \( C_2 \).

Proof: Suppose that M has two distinct submodules \( C_1 \) and \( C_2 \) with \( C_1 \) is closed in M which are isomorphic. Let \( \theta \) be the given isomorphism. No loss of generality, if we assume that \( C_1 \not\subseteq C_2 \), then there is a non-zero element \( x \in C_1 \) and \( x \notin C_2 \). Consider the following two homomorphisms

\[ i_2 \circ \theta : C_1 \rightarrow M \text{ and } i_1 \circ \theta : C_2 \rightarrow M \]
where \( i_1, i_2 \) are the inclusion mapping of \( C_1 \) and \( C_2 \). Since M is a generalized fully stable module, then

\[ i_1 \circ \theta \circ (i_2 \circ \theta)(x) \in C_2 \]
which is a contradiction.

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An R-module M is called Cl-semi-injective, if each Rendomorphism of a closed submodule of M extends to an Rendomorphism of M [10].

Theorem (1.16): Let M be a Cl-semi-injective generalized fully stable R-module and \( T=\text{End}_\alpha(M) \). If \( I=\{\alpha \in T | \ker(\alpha) \text{ is essential in } M\} \), then \( T/I \) is a Von Neumann regular ring.

Proof: For each \( g \in T-I \), there exists a non-zero submodule A of M such that \( A \cap \ker(g)=0 \). By Zorn’s lemma, let B be a maximal submodule of M with the property \( B \cap \ker(g)=0 \). Then \( M_0=B \oplus \ker(g) \) is an essential submodule of M. Now, B being a direct summand and hence is a closed submodule of \( M_0 \). We claim that B is a closed submodule of M. Let N be a submodule of M with \( B \) is essential in N. Then \( B \subseteq M_0 \cap N \), and hence B is an essential submodule of \( M_0 \cap N \). Closeness of B in \( M_0 \) implies that \( B=M_0 \cap N \). Suppose that \( N \cap \ker(g) \neq 0 \), then \( 0 \neq N \cap \ker(g) \cap M_0=B \cap \ker(g) \). which is a contradiction, so \( N \cap \ker(g)=0 \). Maximality of B implies that \( B=N \).

Write \( g_0(=g|_B) : B \rightarrow B \), since M is generalized fully stable, \( \ker(g_0)=B \cap \ker(g)=0 \), thus \( g_0 \) is an R-monomorphism, so \( B=g_0(B) \). By lemma (1.14), \( g_0(B)=B \). Now, the corresponding \( \alpha : g_0(B) \rightarrow g_0(B) \) given by \( \alpha(g_0(b))=b \) is well-defined R-endomorphism. Cl-semi-injective of M implies that there exists \( f \in T \) which is an extension of \( \alpha \). Let
b+y \in M_0 \text{ where } b \in B \text{ and } y \in \ker(g). \ (g-gfg)(b+y)=g(b)-gfg(b)=g(b)-g(b)=0, \text{ then } M_0 \subseteq \ker(g-gfg) , \text{ hence } (g-gfg) \in I. \text{ Thus } T/I \text{ is regular.}

Corollary(1.17): Let M be a CI-semi-injective generalized fully stable R-module and T=\text{End}_R(M). \text{ Then } J(T)=\{ \alpha \in T \mid \ker(\alpha) \text{ is essential in } M \} \text{ and } T/J(T) \text{ is regular.}

Theorem(1.18): Every semi-injective generalized fully stable R-module is continuous.

Proof: Let M be a semi-injective generalized fully stable R-module.
By theorem(1.4), M is quasi-continuous. On the other hand, if we write T=\text{End}_R(M), \text{ then corollary(1.17) implies that } J(T)=\{ \alpha \in T \mid \ker(\alpha) \text{ is essential in } M \} \text{ and } T/J(T) \text{ is regular.} \text{ Thus } M \text{ is a continuous R-module, ([12], proposition(3.15)).}

§2. SEMI-INJECTIVITY VERSUS QUASI-INJECTIVITY
In this section we study semi-injective modules over generalized uniserial rings. First, we recall some concepts and results.

Let \( M = \bigoplus_{\Lambda} M_i \) be a direct sum of R-modules \( M_i \) and \( E(M) = \bigoplus_{\Lambda} E_i \), consider the following set \( K_{ij} = \{ \alpha \in \text{Hom}_R(E_i, E_j) \mid \alpha(N_i) \subseteq N_j \text{ for some essential submodule } N_i(N_j) \text{ of } M_i(M_j) \} \).

The following theorem is proved in [15].

Theorem(2.1): Let R be a noetherian ring and \( M = \bigoplus_{\Lambda} M_i \) be any direct sum of R-modules \( M_i \). Then M is semi-injective if, and only if, \( K_{ij} \subseteq M_j \text{ for each } i, j \in \Lambda \).

An artinian ring R is said to be generalized uniserial, if for every primitive idempotent \( e \) of R, Re has a unique composition series as R-module. These rings were called serial by Eisebud Griffitt [16]. An R-module M of finite composition length is said to be uniserial, if it has a unique composition series. This is equivalent to saying that, all submodule of M are linearly ordered with respect to inclusion.

Theorem(2.2): (Nakayama). Let R be a generalized uniserial ring. Then every R-module is a direct sum of uniserial modules.
Nakayama's theorem says that, any indecomposable module over a generalized uniserial ring is uniserial. Let $M$ and $N$ be two indecomposable semi-injective modules over a generalized uniserial ring $R$ and $E(M)$, $E(N)$ be their injective envelopes. By using corollary(1.3), it is an easy matter to see that $E(M)$, $E(N)$ are indecomposable uniserial $R$-modules. Let $m(E(M), E(N))$ denote the submodule of $E(M)$ which is minimal among the kernels of all $R$-homomorphisms $\sigma$ of $E(M)$ into $E(N)$ with $\sigma(M) \subseteq N$ for some essential submodule $M(\subseteq N)$ of $M(\subseteq N)$. As $E(M)$ is uniserial, $m(E(M), E(N))$ is well-defined and unique. Note that, $m(E(M), E(N))=0$ if, and only if, there is an $R$-monomorphism of $E(M)$ into $E(N)$.

In the following theorem we characterize semi-injective modules over generalized uniserial rings.

**Theorem (2.3):** Let $M$ be a module over a generalized uniserial ring $R$. Then $M$ is semi-injective if, and only if, $M=\bigoplus_{i \in \Lambda} N_i$ where $N_i$ are uniserial modules and $l(N_i) \leq l(N_j) + l(m(E(N_i), E(N_j)))$ for all $i, j \in \Lambda$.

**Proof:** By theorem (2.2), $M=\bigoplus_{i \in \Lambda} N_i$ where the $N_i$'s are uniserial modules. Since $R$ is noetherian, then $E(M)=\bigoplus_{i \in \Lambda} E(N_i)$[17]. For convenience, let as write $E_i$ for $E(N_i)$, then theorem (1.2) implies that $M$ is semi-injective if, and only if, $\sigma(N_i) \subseteq N_j$ for all $\sigma \in K_{ij}$. Now, let $M$ be a semi-injective $R$-module and $\sigma : E_i \rightarrow E_j$ with $\sigma(N_{i \sigma}) \subseteq N_{j \sigma}$ for some essential submodule $N_i(\subseteq N_j)$ of $N_i(\subseteq N_j)$ and $\ker(\sigma) = m(E_{i \sigma}, E_{j \sigma})$. Since $E_i$ is uniserial, then either $N_i \subseteq m(E_i, E_j)$ or $N_i \supseteq (E_i, E_j)$. If $N_i \subseteq m(E_i, E_j)$, then obviously $l(N_i) \leq l(N_j) + l(m(E(N_i), E(N_j)))$. Otherwise, we must have $m(E_i, E_j) \subseteq N_i$, then $N_i / m(E_i, E_j) = \sigma(N_i) \subseteq N_j$ gives $l(N_i) \leq l(N_j) + l(m(E(N_i), E(N_j)))$. Conversely, let $l(N_i) \leq l(N_j) + l(m(E(N_i), E(N_j)))$ for all $i, j \in \Lambda$. For each $\alpha \in K_{ij}$, the minimality implies that $m(E_i, E_j) \subseteq \ker(\alpha)$. Thus using the inequality we immediately get $\alpha(N_i) \subseteq N_j$. Hence $M$ is semi-injective.

We consider the following chain condition on a ring $R$ relative
to a given family of $R$-modules $\{M_{\alpha} | \alpha \in \Lambda \}$ [12].

(*) For every choice of $x \in M_{\alpha}$ ($\alpha \in \Lambda$) and $m_i \in M_{\alpha}$. For distinct $\alpha \in \Lambda$ ($i \in \mathbb{N}$) such that $\text{ann}_R(x) \subseteq \text{ann}_R(m_i)$, the ascending sequence $n_i \geq \cap \text{ann}_R(m_i)$, $(n \in \mathbb{N})$ becomes stationary.

We need the following two results which appear in [12].

Proposition(2.4): The following statements are equivalent for a direct sum decomposition of a module $M = \bigoplus_{\alpha \in \Lambda} M_{\alpha}$.

1. $M$ is quasi-continuous.
2. $\alpha M$ is quasi-injective for every $\beta \in \Lambda \setminus \{\alpha\}$.
3. $\alpha M$ is quasi-injective for all $\alpha, \beta \in \Lambda$ and condition (*) holds.

Proposition(2.5): Let $\{ \alpha M | \alpha \in \Lambda \}$ be a family of quasi-continuous modules. Then the following statements are equivalent.

1. $M = \bigoplus_{\alpha \in \Lambda} M_{\alpha}$ is quasi-continuous.
2. $\beta M$ is quasi-injective for every $\beta \in \Lambda \setminus \{\alpha\}$.

The relation between quasi-injectivity and semi-injectivity was studied in [11]. In fact, we established condition versus semi-injectivity to quasi-injectivity. In particular, we proved that every semi-injective module $M$ over a QEN ring with non-zero socle $S(M)$ is quasi-injective [11]. In this direction we have the following.

Theorem(2.6): Every semi-injective module over a generalized uniserial ring is quasi-injective.

Proof: Let $M$ be a semi-injective module over a generalized uniserial ring $R$. By theorem(2.3), $M = \bigoplus_{i \in I} M_i$ where the $M_i$'s are uniserial modules and $|M_i| \leq |M_j| + |E(M_i)E(M_j)|$ for all $i, j \in I$. Theorem(1.4) implies that $M$ is quasi-continuous and hence $M_i$ is quasi-continuous for each $i \in I$. Thus $M_j$ is $M_i$-injective for every $j \in I - \{i\}$ and condition (*) holds, proposition (2.5). Now, for each $i \in I$, as $M_i$ is uniserial, then $M_i$ is uniform. Thus $M_i$ has non-zero socle. Since $R$ is noetherian, hence $R$ is QEN ring. Then $M_i$ is quasi-injective for each $i$. Proposition (2.4) implies that $M$ is quasi-injective.

The following corollary is a consequence of theorem(2.6) and theorem(1.18).

Corollary (2.7): Let $M$ be a generalized fully stable module over a
generalized uniserial ring R. Then M is a semi-injective R-module if, and only if, M is a continuous R-module.

Theorem(2.8): Any torsion semi-injective module over a Dedekind domain is quasi-injective.

Proof: Let N be a submodule of a torsion semi-injective module over a Dedekind domain R, and σ : N → M be an R-homomorphism. As an application of Zorn’s lemma, we can assume that σ can not be extended to any submodule N of M containing N properly. We claim that N=M. Let x be an element of M with x ∉ N. Now, annR(x) is an essential ideal of R. Let L=annM(annR(x)). Then L is a submodule of M which contains (x). Thus L is a module over a generalized uniserial ring R/annR(x). As L is a stable submodule of M, then L is also a semi-injective R/annR(x)-module[2]. Hence theorem(2.6) implis that, L is a quasi-injective R/annR(x)-module and hence R-module.

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Define λ : N ∩ (x) → L by λ (z)= σ (z) for each z∈N ∩ (x), since L is a stable submodule of M. As (x) ⊆ L, λ can be extended to an R-homomorphism λ * of L. Define σ * : N+(x) → M by σ *(n+rx)= σ (n)+ λ *(rx). Then σ * is well-defined R-homomorphism and is a proper extension of σ. This is a contradiction. Therefore M is quasi-injective.

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