Some Properties Of Cartesian Product Of Two Fuzzy Normed Spaces

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Abstract

In this paper, the concept of the Cartesian Product of two fuzzy normed spaces is presented. Some basic properties and theorems on this concept are proved. The main goal of this paper is to prove that the Cartesian product of two complete fuzzy normed spaces is a complete fuzzy normed space.

Key words: Fuzzy normed space, Cartesian product, Cauchy sequence, complete fuzzy normed space.

1- Introduction

The fuzzy set concepts was introduced in mathematics by K. Menger in 1942 and reintroduced in the system theory by L.A. Zadeh in 1965.

In 1984, Katsaras [1], first introduced the notion of fuzzy norm on linear space, in the same year Wu and Fang [4] also introduced a notion of fuzzy normed space. Later on many other mathematicians like Felbin [2], Cheng and Mordeson [10], Bag and Samanta [12], J. Xiao and X. Zhu [8, 9], Krishna and Sarma [11], Balopoulos and Papadopoulos [13] etc, have given different definitions of fuzzy normed spaces.

J. Kider introduced the definition of fuzzy normed space [7], we use this definition to prove that the Cartesian product of two fuzzy normed spaces is also fuzzy normed space.

The structure of the paper is as follow: In section 2 we present some fundamental concepts. In section 3, the definition of fuzzy normed space appeared [7] is used to prove that the Cartesian product of two fuzzy normed spaces is also fuzzy normed space, then we prove that the Cartesian product of two complete fuzzy normed spaces is complete fuzzy normed space.

2. Preliminaries

In this section, we briefly recall some definitions and preliminary results which are used in this paper.
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Definition 2.1 [7]:
Let $U$ be a vector space over field $K$ ($K=\mathbb{R}$ or $K=\mathbb{C}$). Put $I=[0,1]$ then $\tilde{N}: U \times I \to I$ is said to be a fuzzy norm on $U$ in case for each $u, v \in U$ and $\lambda \in K$ the following conditions hold
(N1) if $\alpha = 0$ then $\tilde{N}(u, \alpha) = 0$.
(N2) if $\alpha \neq 0$ then $\tilde{N}(u, \alpha) = 0$ if and only if $u = 0$.
(N3) $\tilde{N}(\lambda u, \alpha) = |\lambda|\tilde{N}(u, \alpha)$.
(N4) $\tilde{N}(u + v, \alpha) \leq \tilde{N}(u, \alpha) + \tilde{N}(v, \alpha)$.
(N5) if $0 < \sigma \leq \alpha < 1$ then $\tilde{N}(u, \alpha) \leq \tilde{N}(u, \sigma)$ and there exists $0 < \alpha_n < \alpha$ such that $\lim_{n \to \infty} \tilde{N}(u, \alpha_n) = \tilde{N}(u, \alpha)$.

Then $\tilde{N}$ is called fuzzy norm and $(U, \tilde{N})$ is called fuzzy normed space.

Proposition 2.2 [7]:
Let $(U, \| \cdot \|)$ be an ordinary normed space, define $\tilde{N}(u, \alpha) = \frac{1}{\alpha} \|u\|$ for $\alpha > 0$ and $\tilde{N}(u, \alpha) = 0$ for $\alpha = 0$. Then $(U, \tilde{N})$ is a fuzzy normed space.

Example 2.3 [7]:
Let $U = \mathbb{R}$, then $\tilde{N}(u, \alpha) = \frac{1}{\alpha} |u|$ is a fuzzy norm on $\mathbb{R}$ by proposition 2.2 called the usual fuzzy norm.

Definition 2.4 [6]:
Let $A$ and $B$ be any two sets, the Cartesian product is denoted by $A \times B$ and is defined by $A \times B = \{(a, b) | a \in A, b \in B\}$.

Definition 2.5 [3]:
Let $U$ be a universe. A fuzzy set $X$ over $U$ is a set defined by a function $\mu_x$ representing a mapping $\mu_x : U \to [0,1]$.

$\mu_x$ is called a membership function of $X$, and the value $\mu_x(u)$ is called the grade of membership of $u \in U$. Thus a fuzzy set $X$ over $U$ can be represented as follows

$X = \{(\mu_x(u)/u) : u \in U, \mu_x(u) \in [0,1]\}$

Definition 2.6 [5]:
A fuzzy set $x_\alpha$ of a set $S$ is called a fuzzy point if

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y \in S$ and $\alpha \in (0,1]$.

Definition 2.7 [7]:
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A sequence \( \{(u_n, \alpha_n)\} \) of fuzzy points in a fuzzy normed space \((U, \tilde{N})\) converges to \( u_\alpha \in U \) if \( \lim_{n \to \infty} \tilde{N}(u_n - u, \lambda) = 0 \), where \( \alpha, \alpha_n \in (0,1] \) and \( \lambda = \min\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \).

**Definition 2.8 [7]:**
A sequence \( \{(u_n, \alpha_n)\} \) of fuzzy points in a fuzzy normed space \((U, \tilde{N})\) is Cauchy if for any \( \epsilon > 0 \) there is integer \( n_\epsilon > 0 \) such that for all \( m, n > n_\epsilon \), we have
\[
\tilde{N}(u_m - u_n, \lambda) < \epsilon \quad \text{where} \quad \lambda = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}.
\]

3- Cartesian product of two fuzzy normed spaces
In this section, we introduce the definition of the Cartesian product of two fuzzy normed spaces, then we prove that the Cartesian product of two fuzzy normed spaces is also fuzzy normed space. Finally, we prove the completeness of the Cartesian product of two complete fuzzy normed spaces.

**Definition 3.1:**
Let \((U, \tilde{N}_1)\) and \((V, \tilde{N}_2)\) be two fuzzy normed spaces. The Cartesian product of \((U, \tilde{N}_1)\) and \((V, \tilde{N}_2)\) is the product space \((U \times V, \tilde{N})\) where \(U \times V\) is the Cartesian product of the sets \(U\) and \(V\) and \(\tilde{N}\) is a mapping from \((U \times V \times [0,1])\) into \([0,1]\) given by
\[
\tilde{N}((u, v), \alpha) = \tilde{N}_1(u, \alpha) + \tilde{N}_2(v, \alpha), \quad \text{for all} \quad (u, v) \in U \times V, \quad \alpha \in (0,1).
\]

**Theorem 3.2:**
Let \((U, \tilde{N}_1)\) and \((V, \tilde{N}_2)\) be two fuzzy normed spaces then \((U \times V, \tilde{N})\) is a fuzzy normed space.

**Proof:**
Let \((u, v) \in U \times V\) and \(\lambda \in K\).

(N1) if \(\alpha = 0\) then \(\tilde{N}_1(u, \alpha) = 0\) and \(\tilde{N}_2(v, \alpha) = 0\) so
\[
\tilde{N}_1(u, \alpha) + \tilde{N}_2(v, \alpha) = 0 \quad \text{which implies that} \quad \tilde{N}((u, v), \alpha) = 0.
\]

(N2) if \(\alpha \neq 0\) then \(\tilde{N}((u, v), \alpha) = 0 \iff \tilde{N}_1(u, \alpha) + \tilde{N}_2(v, \alpha) = 0.
\]

\(\iff \tilde{N}_1(u, \alpha) = 0 \quad \text{and} \quad \tilde{N}_2(v, \alpha) = 0.
\]

\(\iff (u, \alpha) = 0 \quad \text{and} \quad (v, \alpha) = 0.
\]

\(\iff u = 0 \quad \text{and} \quad v = 0.
\]

\(\iff (u, v) = (0, 0).
\]

(N3) \[
\tilde{N}(\lambda(u, v), \alpha) = \lambda \tilde{N}_1(u, \alpha) + \lambda \tilde{N}_2(v, \alpha)
\]
\[
= \lambda [\tilde{N}_1(u, \alpha) + \tilde{N}_2(v, \alpha)]
\]
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\[ \lambda \mathcal{N}(u, v, \alpha). \]

(\text{N4}) \[ \mathcal{N}\left( ((u, v) + (u_1, v_1)), \alpha \right) \leq \mathcal{N}\left( (u + u_1, v + v_1), \alpha \right) \]
\[ \leq \mathcal{N}_1(u + u_1, \alpha) + \mathcal{N}_2(v + v_1, \alpha). \]
\[ \leq (\mathcal{N}_1(u, \alpha) + \mathcal{N}_2(v, \alpha)) + (\mathcal{N}_1(u_1, \alpha) + \mathcal{N}_2(v_1, \alpha)). \]
\[ = \mathcal{N}\left( (u, v), \alpha \right) + \mathcal{N}\left( (u_1, v_1), \alpha \right). \]

(\text{N5}) if \( 0 < \sigma \leq \alpha < 1 \) then \( \mathcal{N}_1(u, \alpha) \leq \mathcal{N}_1(u, \sigma) \) and \( \mathcal{N}_2(v, \alpha) \leq \mathcal{N}_2(v, \sigma) \), so \( \mathcal{N}\left( (u, v), \alpha \right) \leq \mathcal{N}\left( (u, v), \sigma \right) \) and there exist
\[ 0 < \alpha_n < \alpha \] such that \( \lim_{n \to \infty} \mathcal{N}_1(u, \alpha_n) = \mathcal{N}_1(u, \alpha) \) and
\[ \lim_{n \to \infty} \mathcal{N}_2(v, \alpha_n) = \mathcal{N}(v, \alpha) \] which implies that
\[ \lim_{n \to \infty} \mathcal{N}\left( (u, v), \alpha_n \right) = \mathcal{N}\left( (u, v), \alpha \right). \]

Thus \( (U \times V, \mathcal{N}) \) is fuzzy normed space.

**Proposition 3.3**

If \( \{(u_n, \alpha_n)\} \) is a sequence of fuzzy points in the fuzzy normed space \( (U, \mathcal{N}_1) \) converges to \( u_\alpha \) in \( U \) and \( \{(v_n, \alpha_n)\} \) is a sequence of fuzzy points in the fuzzy normed space \( (V, \mathcal{N}_2) \) converges to \( v_\alpha \) in \( V \) then
\[ \{(u_n, v_n), \alpha_n\} \] is a sequence in \( U \times V \) converges to \( (u_\alpha, v_\alpha) \) in \( (U \times V, \mathcal{N}) \) where \( \alpha = \min\{\alpha_n, n \in N\} \).

**Proof:**

To conclude that sequence \( \{(u_n, v_n), \alpha_n\} \) in \( U \times V \) converges to \( (u_\alpha, v_\alpha) \) we show that \( \lim_{n \to \infty} \mathcal{N}\left( (u_n, v_n) - (u, v), \lambda \right) = 0 \).

By theorem 3.2, \( (U \times V, \mathcal{N}) \) is a fuzzy normed space. Since
\[ (u_\alpha, \alpha_n) \to u_\alpha \] and \( (v_\alpha, \alpha_n) \to v_\alpha \), so \( \lim_{n \to \infty} \mathcal{N}_1(u_n - u, \lambda_1) = 0 \) and \( \lim_{n \to \infty} \mathcal{N}_2(v_n - v, \lambda_2) = 0 \).

So \( \lim_{n \to \infty} \mathcal{N}\left( (u_n, v_n) - (u, v), \lambda \right) = \lim_{n \to \infty} \mathcal{N}_1(u_n - u, \lambda) + \lim_{n \to \infty} \mathcal{N}_2(v_n - v, \lambda) = 0 + 0 = 0 \).

Thus \( \{(u_n, v_n), \alpha_n\} \) converges to \( (u_\alpha, v_\alpha) \).

**Proposition 3.4**

If \( \{(u_n, \alpha_n)\} \) is Cauchy sequence in \( (U, \mathcal{N}_1) \) and \( \{(v_n, \alpha_n)\} \) is Cauchy sequence in \( (V, \mathcal{N}_2) \) then \( \{(u_n, v_n), \alpha_n\} \) is Cauchy sequence in \( (U \times V, \mathcal{N}) \).
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Proof:
By theorem 3.2 , \((U \times V , \tilde{N})\) is a fuzzy normed space . since \(\{(u_n , \alpha_n )\}\) and \(\{(v_n , \alpha_n )\}\) are Cauchy sequences then for each given \(\varepsilon > 0\) there is a positive constant \(n_0\) such that ,
\[\tilde{N}_1(u_m - u_n , \lambda_1) < \frac{\varepsilon}{2} \quad \text{and} \quad \tilde{N}_2(v_m - v_n , \lambda_2) < \frac{\varepsilon}{2}\]
for every \(m, n > n_0\)

Now for each \(m, n > n_0\)
\[\tilde{N}((u_m , v_m) - (u_n , v_n), \lambda) = \tilde{N}((u_m - u_n , v_m - v_n), \lambda) . \]
\[= \tilde{N}_1(u_m - u_n , \lambda_1) + \tilde{N}_2(v_m - v_n , \lambda_2) . \]
\[= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon . \]

Thus \(\{(u_n , v_n), \alpha_n\}\) is Cauchy sequence in \((U \times V , \tilde{N})\).

Definition 3.5:
Let \((U , \tilde{N})\) be a fuzzy normed space , then a fuzzy normed space in which every fuzzy Cauchy sequence \(\{(u_n , \alpha_n)\}\) is convergent is said to be complete.

Theorem 3.6
If \((U , \tilde{N}_1)\) and \((V , \tilde{N}_2)\) are complete fuzzy normed spaces then Cartesian product \((U \times V , \tilde{N})\) is a complete fuzzy normed space .

Proof:
Let \(\{(u_n , v_n), \alpha_n\}\) be a Cauchy sequence in \(U \times V\) that is for any given \(\varepsilon > 0\) there is \(n_0\) such that \(\tilde{N}\left((u_m , v_m) - (u_n , v_n)\right) , \lambda < \varepsilon\) which implies that
\[\tilde{N}_1(u_m - u_n , \lambda_1) + \tilde{N}_2(v_m - v_n , \lambda_2) < \varepsilon \quad \text{where} \quad \lambda = \min\{\lambda_1, \lambda_2\} \]
so that \(\tilde{N}_1(u_m - u_n , \lambda_1) < \varepsilon\) and \(\tilde{N}_2(v_m - v_n , \lambda_2) < \varepsilon\) that is \(\{(u_n , \alpha_n)\}\) is Cauchy in \((U , \tilde{N}_1)\) and \(\{(v_n , \alpha_n)\}\) is Cauchy in \((V , \tilde{N}_2)\) , but \((U , \tilde{N}_1)\)
and \((V , \tilde{N}_2)\) are complete fuzzy normed spaces , so there is \(u_\alpha\) in \(U\) and \(v_\alpha\) in \(V\) such that \(\{(u_n , \alpha_n)\}\) converges to \(u_\alpha\) and \(\{(v_n , \alpha_n)\}\) converges to \(v_\alpha\) that is \(\lim_{n \to \infty} \tilde{N}_1(u_n - u , \lambda_1) = 0\) and \(\lim_{n \to \infty} \tilde{N}_2(v_n - v , \lambda_2) = 0\) .

Now ,
\[\lim_{n \to \infty} \tilde{N}\left(((u_n , v_n) - (u , v))\right) = \]
\[\lim_{n \to \infty} \tilde{N}_1(u_n - u , \lambda_1) + \lim_{n \to \infty} \tilde{N}_2(v_n - v , \lambda_2) = 0 + 0 = 0\]
Thus \(\{(u_n , v_n), \alpha_n\}\) converges to \((u_\alpha , v_\alpha)\) in \(U \times V\) , therefore \((U \times V , \tilde{N})\)
is a complete fuzzy normed space.
Theorem 3.7
If \((U \times V, \tilde{N})\) is a fuzzy normed space, then \((U, \tilde{N}_1)\) and \((V, \tilde{N}_2)\) are fuzzy normed spaces by defining \(\tilde{N}_1(u, \alpha) = \tilde{N}((u, 0), \alpha)\) and \(\tilde{N}_2(v, \alpha) = \tilde{N}((0, v), \alpha)\).

Proof:
Let \(u \in U\) and \(v \in V\).

(N1) if \(\alpha = 0\) then \(\tilde{N}((u, 0), \alpha) = 0 \Rightarrow \tilde{N}_1(u, \alpha) = 0\)

(N2) if \(\alpha \neq 0\) then \(\tilde{N}_1(u, \alpha) = 0 \Leftrightarrow \tilde{N}((u, 0), \alpha) = 0\)

\[\Leftrightarrow (u, 0) = 0\]

\[\Leftrightarrow u = 0\]

(N3) \(\tilde{N}_1(\gamma u, \alpha) = \tilde{N}((\gamma u, 0), \alpha) = |\gamma| \tilde{N}(u, 0, \alpha) = |\gamma| \tilde{N}_1(u, \alpha)\).

(N4) \(\tilde{N}_1(u + u_1, \alpha) = \tilde{N}((u + u_1, 0), \alpha)\)

\[\leq \tilde{N}((u, 0), \alpha) + \tilde{N}((u_1, 0), \alpha)\]

\[= \tilde{N}_1(u, \alpha) + \tilde{N}_1(u_1, \alpha)\).

(N5) if \(0 < \sigma \leq \alpha < 1\) then \(\tilde{N}((u, 0), \alpha) \leq \tilde{N}((0, 0), \sigma)\) that is \(\tilde{N}_1(u, \alpha) \leq \tilde{N}_1(u, \sigma)\).

Then there exists \(0 < \alpha_n < \alpha\) such that \(\lim_{n \to \infty} \tilde{N}_1(u, \alpha_n) = \tilde{N}_1(u, \alpha)\).

Thus \((U, \tilde{N}_1)\) is a fuzzy normed space.

Similarly we can prove that \((V, \tilde{N}_2)\) is a fuzzy normed space.

Theorem 3.8:
If \((U \times V, \tilde{N})\) is a complete fuzzy normed space, then \((U, \tilde{N}_1)\) and \((V, \tilde{N}_2)\) are complete fuzzy normed spaces.

Proof:
\((U, \tilde{N}_1)\) and \((V, \tilde{N}_2)\) are fuzzy normed spaces by theorem 3.7.

Let \(\{(u_n, \alpha_n)\}\) be a Cauchy sequence in \((U, \tilde{N}_1)\) then \(\{(u_n, 0), \alpha_n)\}\) is a Cauchy sequence in \(U \times V\). But \(U \times V\) is complete fuzzy normed space, that is there is \((u_\alpha, 0)\) in \(U \times V\) such that \(\{(u_n, 0), \alpha_n)\}\) converges to \((u_\alpha, 0)\).

Now, \(\lim_{n \to \infty} \tilde{N}_1(u_n - u, \alpha) = \lim_{n \to \infty} \tilde{N}((u_n - u, 0), \alpha) = 0\)

That is \((U, \tilde{N}_1)\) is a complete fuzzy normed space. Similarly we can prove that \((V, \tilde{N}_2)\) is a complete fuzzy normed space.
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References
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الخلاصة

في هذا البحث تم تقديم مفهوم الضرب الديكارتي لفضائين معياريين (قياسات)، وتم تفسير بعض النتائج (الخصائص) الأساسية والمبرهنات حول هذا المفهوم تم برهانها. الهدف الرئيسي لهذا البحث هو برهان أن الضرب الديكارتي لفضائين معياريين ضبابيين تامين هو فضاء معياري ضبابي تام.