Abstract
In this paper, we study and deal with the fundamental bornological constructions for bornology defines on the space of entire function all element represented by dirichlet series, such as bornological subspace, product bornologies and Quotient bornologies with adding some properties to them.

1. Introduction
The space of entire functions over the complex field $\mathbb{C}$ was introduced by [5] who defined a metric on this space by introducing a real-valued map on it. In (1971), H.Hogbe-Nlend introduced the concepts of bornology on a set. In (1981) M.D. Patwardhan extended this idea to a space of entire functions. However, we shall study the bornological constriction of the space which all element represented by diirschlet series.

In this paper, we consider the space of entire function $\alpha(s)$ represented by Dirichlet series is

$$\alpha(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} = a_1 e^{s\lambda_1} + a_2 e^{s\lambda_2} + \cdots + a_n e^{s\lambda_n} \quad (1)$$

Where $a_i \in \mathbb{C}$, $i \geq 1$, $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geq 0$, $\lim_{n \to \infty} \lambda_n = \infty$, $s = \sigma + it$ ($\sigma, t$ real variables), and $\{a_n\}$ is any sequence of complex number set?

For an entire function $\alpha(s)$, define the number $|\alpha|$ by

$$|\alpha| = \text{l.u.b.} \left| a_n \right| \frac{1}{\lambda_n}, n \geq 1 \quad (2)$$

It is easily verified that $|\alpha|$ satisfies the following conditions…(3)

(i) $|\alpha| \geq 0$ and $|\alpha| = 0$ if and only if $\alpha = 0$, the identically zero function;

(ii) $|\alpha + \beta| \leq |\alpha| + |\beta|$

(iii) $|\omega\alpha| \leq A(\omega)|\alpha|$ Where $A(\omega) = \max (1, |\omega|), \omega$ being any complex number
It follows from (i) and (ii) of (3) that \( |\alpha - \beta| \) defines a distance in the class of entire function.

Then \( \Gamma = \left\{ \alpha : \alpha = \sum_{n=0}^{\infty} a_n \exp(s\lambda_n), a_n \in \mathbb{C}, \lambda_n \to \infty, \lim_{n \to \infty} \left| a_n \right| = 0 \right\} \), the vector space of all entire function all element represented by Dirichlet series.

Lister we define a bornology of the space \( \Gamma \).

The aim of the present paper is study and deals with the fundamental Bornological constructions of \( \Gamma \) and with adding some properties to them, such as Bornological subspace, product bornologies and Quotient bornologies.

**Keywords:** Bornological spaces, Entire functions, Dirichlet series

**2. Bornological Spaces**

In this section we recall several basic notions from the theory of Bornological linear spaces.

**(2.1) Definition**

A Bornology \( B \) on a set \( X \) is a family of subsets of \( X \) such that \( B \) is a covering of \( X \); finite unions of elements of \( B \) are in \( B \); any subset of an element of \( B \) is also in \( B \). The elements of \( B \) are called bounded sets.[1]

**(2.2) Definition**

A Bornological linear space is a linear space over the field \( K \) (the real or complex field) together with a Bornology on underlying set of vectors such that the sum of vectors and the product of elements of \( K \) by vectors are bounded operations, i.e. the sets \( A + B \) and \( C \cdot B \) are bounded sets whenever \( A \) and \( B \) are bounded subsets of \( X \) and \( C \) is a bounded subset of \( K \) by [2]

**(2.3) Definition**

Let \( (X, B) \) be a Bornological space and let \( Y \subseteq X \). Then the collection \( B_Y = \{ B \cap Y : B \in B \} \) is a Bornology on \( Y \) and the Bornological space \( (Y, B_Y) \) is called relative Bornology on \( Y \).[3]

**(2.4) Definition**

Let \( (X_i, B_i) \), \( i \in I \), be a family of Bornological space indexed by a non-empty set \( I \) and let \( X = \prod_{i \in I} X_i \) be the product of the sets \( X_i \). For every \( i \in I \), let \( p_i : X \to X_i \) be the canonical projection then. The product Bornology on \( X \) is the initial Bornology on \( X \) for the maps \( p_i \). The set \( X \),
endowed with the product bornology is called the Bornological product of the space \((X_i, B_i)\).[3]

**Definition**

Let \((X, B)\) be a Bornological space and \(\theta: X \to Y\) be a map of \(X\) onto a set \(Y\). Then \([B_{\theta}] = \{ \theta(B) : B \subseteq X \text{ bounded} \}\) is the quotient bornology of \(B\) under \(\theta\). \(Y\) is called a quotient space of \(X\), \(\theta\) is called a quotient map (Final Bornology). [3]

**Definition**

A base of a Bornology \(B\) on \(X\) is any subfamily \(B_0\) of \(B\) such that every element of \(B\) is contained in an element of \(B_0\). [1]

3. Bornological Constriction of the space \(\Gamma\).

In this section study the fundamental Bornological constrictions of the space \(\Gamma\) which all element represent by Dirichlet series such as Bornological subspace, product Bornologies and Quotient Bornologies.

3.1 Bornological Subspace of the Space \(\Gamma\).

All Definition given in (2) can be applied for the special case where

\[ X = \Gamma \text{ and } \Gamma = \left\{ \alpha : \alpha = \sum_{n=0}^{\infty} a_n \exp(s\lambda_n), a_n \in \mathbb{C}, \lim_{n \to \infty} \left| a_n \right| \lambda_n = 0 \right\}, \text{ the vector space of all entire function all element represented by Dirichlet series} \]

**Theorem**

Let \(\Gamma\) be a vector space of all entire functions over the complex field \(\mathbb{C}\), let \((\Gamma, B)\) be a Bornological vector space and \(Y\) be a vector subspace of \(\Gamma\). Then the collection \(B_y = \{ B \cap Y : B \in B \} \) is a vector bornology on \(Y\).

**Proof:** (by definition of bornology) \(B_y\) is a bornology on \(Y\).

To prove \(B_y\) is a vector bornology on \(Y\), \(\forall A, B \in B_y\) and \(\lambda \in \mathbb{C}\)

Let \(A = V \cap Y\), \(B = U \cap Y\) where \(\{ U, V \in B \}\)

\[
\left\{ \begin{array}{l}
\therefore A \subseteq V, \because V \in B, \therefore A \in B \\
\therefore B \subseteq U, \therefore U \in B, \therefore B \in B
\end{array} \right\}
\]

I.e. every element of \(B_y\) is an element of \(B\).
BORNOLOGICAL ON THE SPACE ALL ELEM IN REPRESENTED BY DIRICHLET SERIES

ANWAR NOOR AL-DEEN AL-SALIHI , SAAD QASSIM FLEH

Since \( B \) is a vector bornology on \( \Gamma \), then \( \lambda B, A, \lambda A \in B \)

\[
\begin{cases}
A + B \\
\lambda A
\end{cases}
\]

\[
\cup \lambda A, |\lambda| < 1
\]

\[\because Y \text{ be vector subspace of } \Gamma: \left\{ \begin{array}{l}
A + B \\
\lambda A
\end{array} \right\} \subseteq Y, \text{then } \left\{ \begin{array}{l}
A + B \\
\lambda A
\end{array} \right\} \in B_Y
\]

I.e. \( B_Y \) is stable under vector addition, homothetic transformation and circled hull.

Then \((Y, B_Y)\) is a Bornological vector subspace of \((\Gamma, B)\).

(3.1.2) Remark

It is clear that every bounded subset of \( Y \) is also a bounded subset of \( \Gamma \), i.e. \( B_Y \subseteq B \), and every subspace of discrete space is discrete space.

(3.1.3) Proposition

Let \( W \subseteq Y \subseteq \Gamma \) Then if \( B_W \) is the vector bornology on \( W \) w.r.t \((\Gamma, B)\)

and \( B_w \) is the vector bornology on \( W \) w.r.t \((Y, B_Y)\). Then \( B_W = B_w \)

Proof: Let the bornology \( B (\Gamma) \) given to \( W \) from \( \Gamma \). And from \( Y \) to be \( B (Y) \) (Since every bounded subset of \( Y \) is also a bounded subset of \( X \))

then it is clear that \( B (Y) \subseteq B (\Gamma) \) \( \ldots \) (1)

Let \( V \in B (\Gamma) \), then \( \exists \gamma \subseteq \Gamma \) such that

\[ V = W \cap \gamma = (W \cap Y) \cap \gamma = W \cap (Y \cap \gamma), V \subseteq \Gamma, \text{ then } Y \cap V = V^* \subseteq Y \]

Then \( V = W \cap V^* \), \( V^* \subseteq Y \) and \( V \in B (Y) \)

Then \( B (\Gamma) \subseteq B (Y) \) \( \ldots \) (2) From (1) and (2), then \( B_W = B_w \)

(3.1.4) Example

Let \( \Gamma \) denotes the vector space of all entire function all element represented by Dirichlet series, and \( Y \subseteq \Gamma \) such that

\[ Y = \left\{ a = \sum_{k=0}^{\infty} \frac{(ke^{i})^{2n+1}}{(2n+1)!} = \sin(ke^{i}) : \lim_{n \to \infty} \frac{k^{2n+1}}{(2n+1)!} = 0, k \text{ is real} \right\} \]

and let \( B = \{ B : B \subseteq \Gamma \} \),

\[ B = \left\{ \alpha : \| \alpha \| = \sup \left\{ k \cdot \left| \frac{k^{2n+1}}{(2n+1)!} \right|^{1/n}, n \geq 1 \right\} \right\}, \text{ then } B_Y = \{ A = B \cap Y : B \in B \}, \]

\[ A = \alpha(s) : \| \alpha(s) \| = \sup \left\{ \alpha \left( \frac{1}{a_n^{1/n}} \right) \right\} \]
Now to define the base of the subspace of Bornological vector space \((\Gamma, \mathcal{B})\). We denote by \(B_r\) the set of all \(\{x(s) : \|x(s)\| \leq r\}\). Then the family \(B_0 = \{B_r : r = 1, 2, \ldots\}\) forms a base for a bornology \(\mathcal{B}\) on \(\Gamma\), then the base of the subspace is \(B_0' = \{A = B_r \cap Y : B_r \in B_0\}\) such that \(A = \{x(s) : \|x(s)\| \leq r, r_1 \leq r\}\)

Where \(r\) integer.

\[3-2\text{ Product Bornology of the Space } \Gamma\]

\[(3.2.1)\text{ Definition}\]

Let \((\Gamma_i, B_i)_{i \in I}\) be a family of Bornological vector space indexed by a non-empty set \(I\) and let \(\Gamma \equiv \prod_{i \in I} \Gamma_i = \left\{ \prod_{i \in I} x_i : \forall x_i \in \Gamma_i, i \in I \right\}\) be the product of the vector space \(\Gamma_i\). For every \(i \in I\), let \(p_i : \Gamma \rightarrow \Gamma_i\) be the canonical projection then the product bornology on \(\Gamma\) is the coarsest bornology on \(\Gamma\) for which each map \(p_i\) is bounded. I.e. the product bornology is \(B = \left\{ \prod_{i \in I} B_i : \forall B_i \in B_i, i \in I \right\}\),

\(B_i = \{\alpha(s) : \|\alpha(s)\| = \ell_u p_i \left[\alpha_i \right]^{\frac{1}{r_i}}\}\)

Now to define the base of the product bornologies of Bornological vector space \((\Gamma_i, B_i)_{i \in I}\).

\[(3.2.2)\text{ Definition}\]

The product bornology on \(\Gamma\) has a base consisting of sets of the form \(B = \prod B_{0i}\) where \(B_{0i} \in B_{0i}\) for all \(i \in I\) where \(B_{0i}\) is a base for \(\Gamma_i\).

\[\prod_{i \in I} B_i = \left\{ \prod_{i \in I} \alpha_i : \|\alpha_i\| \leq r, r_1 = 1, 2, \ldots, i \in I \right\}\]

\[3-3\text{ Quotient Bornology of the space } \Gamma\]

\[\text{Definition}(3.2.3)\]

Let \(F\) be a linear subspace of a Bornological vector space \(\Gamma\), the linear Bornological quotient space is the quotient space \(\Gamma/F\). with the quotient Bornology on \(E/F\) such that \(i.e B = \{ \theta(B) : \theta(B) = B+F, B\text{ is bounded in }\Gamma \}

\[\text{References}\]


Abdul Hussein and Al-shaibani, M. Entire Harmonic Functions and \[\text{Bornological Space}\], ph. D Thesis Indian Institute of Technology Roorkee Indian.

BORNLOGICAL ON THE SPACE ALL ELEMIN
REPRESENTED BY DIRICHLET SERIES ...........................
ANWAR NOOR AL-DEEN AL-SALIHI, SAAD QASSIM FLEH

Publishing Company Netherlands.

by Dirichled series, Collectania Mathematical", 19(3)203-216.


حوال التراكيب البرنولوجية لفضاء كل عناصره ممثلة

 بواسطة متسلسلة داشليت

انوار نور الدين عمران

تدريسة في كلية الهندسة –قسم هندسة الحاسبات –جامعة ديالي

الستخلص

في هذا البحث قمنا بدراسة تراكيب برنولوجية أساسية لبرنولوجي معروف على فضاء الدوال
الكلية التي كل عناصره ممثلة بواسطة سلسلة داشليت إضافة بعض الخواص لها مثل
لفضاء البرنولوجي الجزئي، وضاءة الجداء البرنولوجي، وفضاء القسمة