

Sixth Order Finite Difference In Two Dimensions With Some Examples

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Abstract

A nine-points sixth – order accurate compact finite difference method for solving Laplace equation in two dimensional is present. The efficiency of the method is validated by its application to the test problems which have exact solutions. Numerical results show that this sixth order scheme has the expected accuracy. The explicit numerical solution of the parabolic equation is given, In addition, can be applicable to domain with some obstacle, as well, improve the solution of Poisson equation by taking more terms from the right hand side. All the computations are solved by the iterative method with personal computer.

Keywords: Six order finite difference, Laplace equation, parabolic equation, Poisson equation.

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(1) Introduction

The partial differential equation $\nabla u = 0$ is Laplace equation governs many important physical phenomena and its applications, numerical solution of steady state of heat transfer[10], mobile robot [2], Laplace equation using fuzzy data [9], frequency space two dimensions scalar extrapolator using finite difference operator [4], and so on. For the traditional finite difference methods, in order to increase the order of accuracy of approximations, by increasing the grid points. Dirichlet boundary conditions are used. Assume that the solution $u(x, y)$ have sufficiently smooth and the necessary continuous partial derivatives up to certain orders. The (PDE) of Laplace equation in two dimensional can be written in the form of

$$u_{xx}(x, y) + u_{yy}(x, y) = 0, \quad (x, y) \in \Omega$$

Where Ω is a rectangular domain, or union of rectangular domains, with suitable boundary conditions defined on $\partial\Omega$.

(2) An Extension Method for sixth order Finite Difference in Two Dimensions

In this section, the method is developed similar to sixth order accurate approximation [3], [7], [11]. The scheme of two dimensional rectangular domain to have nine-points stencil which is equal spaces in both directions x and y . Laplace equation given by:

$$u_{xx}(x, y) + u_{yy}(x, y) = 0, \quad (x, y) \in \Omega \dots\dots\dots (1)$$

The central difference finite difference method is written by:

$$\delta_x^2 u_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \text{ and } \delta_y^2 u_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}, \quad \text{where}$$

$$u_{ij} = u(x_i, y_j) \dots\dots (2)$$

By Taylor expansions at the point (x_i, y_j) , we have

$$u_{i+1,j} = u_{ij} + hu_x + \frac{h^2}{2!}u_{xx} + \frac{h^3}{3!}u_{xxx} + \frac{h^4}{4!}u_{xxxx} + \frac{h^5}{5!}u_{xxxxx} + \frac{h^6}{6!}u_{xxxxxx} + \dots$$

$$u_{i-1,j} = u_{ij} - hu_x + \frac{h^2}{2!}u_{xx} - \frac{h^3}{3!}u_{xxx} + \frac{h^4}{4!}u_{xxxx} - \frac{h^5}{5!}u_{xxxxx} + \frac{h^6}{6!}u_{xxxxxx} - \dots$$

Add the above equations we get

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} = \frac{\partial^2 u}{\partial x^2} + \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4} + \frac{h^4}{360} \frac{\partial^6 u}{\partial x^6} + \frac{h^6}{20160} \frac{\partial^8 u}{\partial x^8} + \dots$$

or

$$\delta_x^2 u_{i,j} = \frac{\partial^2 u}{\partial x^2} + \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4} + \frac{h^6}{360} \frac{\partial^6 u}{\partial x^6} + O(h^6) \dots\dots\dots (3)$$

In similar manner

$$\delta_y^2 u_{i,j} = \frac{\partial^2 u}{\partial y^2} + \frac{h^2}{12} \frac{\partial^4 u}{\partial y^4} + \frac{h^6}{360} \frac{\partial^6 u}{\partial y^6} + O(h^6) \dots\dots\dots (4)$$

Substitute (3) and (4) in (1) we obtain

$$\delta_x^2 u_{i,j} + \delta_y^2 u_{i,j} + T_{i,j} = 0 \dots\dots\dots(5)$$

Where $u_{i,j} = u(x_i, y_j)$, and

$$T_{i,j} = -\frac{h^2}{12} \left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \right)_{i,j} - \frac{h^4}{360} \left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} \right)_{i,j} + O(h^6) \dots\dots\dots (6)$$

Using the following appropriate derivatives of equation (1):

$$\frac{\partial^4 u}{\partial x^4} = -\frac{\partial^4 u}{\partial x^2 \partial y^2} \text{ and } \frac{\partial^4 u}{\partial y^4} = -\frac{\partial^4 u}{\partial y^2 \partial x^2} \dots\dots\dots (7)$$

Substitute in (6) , we get:

$$T_{i,j} = -\frac{h^2}{12} \left(-2 \left[\frac{\partial^4 u}{\partial x^2 \partial y^2} \right]_{i,j} \right) - \frac{h^4}{360} \left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} \right)_{i,j} + O(h^6) \dots \dots \dots (8)$$

This derivation of $O(h^6)$ scheme, the fourth order approximation of the term $\frac{\partial^4 u}{\partial x^2 \partial y^2}$ in equation (8) is given by:

$$\left[\frac{\partial^4 u}{\partial x^2 \partial y^2} \right]_{i,j} = \delta_x^2 \delta_y^2 u_{i,j} - \frac{h^2}{12} \left(\frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4} \right)_{i,j} + O(h^4) \dots \dots \dots (9)$$

Substituting equation (9) in (8) , we get :

$$T_{i,j} = \frac{h^2}{12} (2\delta_x^2 \delta_y^2 u_{i,j}) - \frac{h^4}{360} \left(\frac{\partial^6 u}{\partial x^6} + 5 \frac{\partial^6 u}{\partial x^4 \partial y^2} + 5 \frac{\partial^6 u}{\partial x^2 \partial y^4} + \frac{\partial^6 u}{\partial y^6} \right)_{i,j} + O(h^6) \dots \dots (10)$$

getting a compact sixth- order approximation requires compact expressions of the four derivatives of order six in (10), which can be done by further differentiating equation (1), that is :

$$\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} = - \left(\frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4} \right) \dots \dots \dots (11)$$

Substituting equations (7) into equation (11) gives us:

$$T_{i,j} = \frac{h^2}{12} (2\delta_x^2 \delta_y^2 u_{i,j}) - \frac{h^4}{360} \left(4 \left[\frac{\partial^4 u}{\partial x^2 \partial y^2} \right]_{i,j} \right) + O(h^6) \dots \dots \dots (12)$$

The compact sixth-order approximation of the Laplace equation in two dimensional can be obtained as:

$$\frac{h^2}{6} \delta_x^2 \delta_y^2 U_{i,j} + (\delta_x^2 + \delta_y^2) U_{i,j} = 0 \dots \dots \dots (13)$$

Where $U_{i,j} = u_{i,j}$ is the discrete approximation to $u_{i,j}$ satisfying the discrete formulation of equation (1) that is, $u_{i,j} = U_{i,j} + O(h^6)$

We can express the equation (13) in the form of

$$K_{20}U_{i,j} + K_{21}H + K_{22}X = 0 \quad \dots\dots\dots (14),$$

Where

$$H = U_{i+1,j} + U_{i-1,j} + U_{i,j} + 1 + U_{i,j-1} \quad \dots\dots\dots (15)$$

$$X = U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1} + U_{i-1,j-1} \quad \dots\dots\dots (16)$$

$$K_{20} = -\frac{10}{3}, \quad K_{21} = \frac{2}{3}, \quad \text{and} \quad K_{22} = \frac{1}{6} \quad \dots\dots\dots (17) .$$

May write equation (15) in the form:

$$-\frac{10}{3}U_{i,j} + \frac{2}{3}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}) + \frac{1}{6}(U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i-1,j-1}) = 0 \quad \dots\dots\dots (18) \quad \text{or}$$

We can use more terms as derived in [8], we have the following correspondence formula:

$$h^2\nabla^2 = \frac{1}{6}(2H + X) - \frac{1}{420}(13H^2 + 16HX + 6X^2) + \frac{1}{630}(2H^3 + H^2 + 4HX^2) - \dots\dots(19)$$

$$6h^2\nabla^2 = (2H + X) - \frac{1}{70}(13H^2 + 16HX + 6X^2) + \frac{1}{15}(2H^3 + H^2 + 4HX^2) - \dots\dots(20)$$

In actual solution of Laplace equation using only one term of the right hand side of equation (20).

(3) Examples:

(1) Laplace Equation

Laplace equation it has many applications, heat transfer, potential energy, path planning for Mobile Robot [2], Laplace equation using Fuzzy Data [8], and so on ...[5,6,11,12,13]. We applying the first term of equation (20) to some example and we compare the results with exact solution.

Example (1):

The Laplace equation $\Delta u = 0$, in the rectangular domain $0 \leq x \leq 1$, $0 \leq y \leq 1$, where the boundary condition and the exact solution is given by $u(x, y) = \exp(-2x) \sin(2y)$.

Solution

We solve this example by three methods one using five points diagonally, the second five points cross section, and the third one using 9-points implements equation (20).

| Step length | Max error using five points diagonally | Max error using five points Cross section | Max error using nine points |
|-------------|--|---|-----------------------------|
| 0.25 | 8.552833E-03 | 3.623592326E-03 | 4.722111879E-07 |
| 0.2 | 5.782082E-03 | 2.581170344E-03 | 1.355933595E-07 |
| 0.1 | 1.358132E-03 | 6.612918926E-04 | 2.117365338E-09 |
| 0.05 | 3.375636E-04 | 1.676534456E-04 | 3.333783249E-11 |

(2) Parabolic Equation $U_t = C^2 \nabla^2 U$.

By differentiation and substitution from the equation $U_t = C^2 \nabla^2 U$ we get the sequence [8]:

$$U_{t^m} = C^{2m} \nabla^{2m} U_t \text{ for } m = 1, 2, \dots$$

Then by Taylor's series

$$\begin{aligned}
 U(x, y, t + k) - U(x, y, t) &= kU_t + \frac{k^2 U_{tt}}{2!} + \frac{k^3 U_{ttt}}{3!} + \dots \\
 &= \frac{1}{6} h^2 \nabla^2 U + \frac{1}{36} h^4 \nabla^4 U + \dots
 \end{aligned}$$

Where we have set $k = h^2 / 6C^2$. Now for $h^2 \nabla^2 U$ we substitute the expression in terms of differences given by equation (20), collect differences of like orders, and obtain finally:

$$U(x, y, t + k) - U(x, y, t) = \frac{1}{36} (4M + N)U(x, y, t) + \dots \text{ where}$$

$$\begin{aligned}
 M &= (U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}) \\
 N &= (U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i-1,j-1})
 \end{aligned}$$

In actual computation we drop all terms on the right in equation except the first term and represent the resulting equation by the stencil

$$U(x, y, t + k) = \frac{1}{36} \begin{bmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{bmatrix} U(x, y, t).$$

Example (2) Solve the equation $U_t = C^2 \nabla^2 U$ with the $U = 0$ on the boundary of the square of side $6h$, and the initial value of U given by $U = xy(6h - x)(6h - y) / h^4$.

| K=0 | | | | | | |
|-----|----|----|----|----|----|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 25 | 40 | 45 | 40 | 25 | 0 |
| 0 | 40 | 64 | 72 | 64 | 40 | 0 |
| 0 | 45 | 72 | 81 | 72 | 45 | 0 |
| 0 | 40 | 64 | 72 | 64 | 40 | 0 |
| 0 | 25 | 40 | 45 | 40 | 25 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| K=1 | | | | | | |
|-----|-------|-------|-------|-------|-------|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 21.78 | 35.78 | 40.44 | 35.78 | 21.78 | 0 |
| 0 | 35.78 | 58.78 | 66.44 | 58.78 | 35.78 | 0 |
| 0 | 40.44 | 66.44 | 75.11 | 66.44 | 40.44 | 0 |
| 0 | 35.78 | 58.78 | 66.44 | 58.78 | 35.78 | 0 |
| 0 | 21.78 | 35.78 | 40.44 | 35.78 | 21.78 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| K=2 | | | | | | |
|-----|-------|-------|-------|-------|-------|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 19.26 | 32.19 | 36.57 | 32.19 | 19.26 | 0 |
| 0 | 32.19 | 53.78 | 61.11 | 53.78 | 32.19 | 0 |
| 0 | 36.57 | 61.11 | 69.44 | 61.11 | 36.57 | 0 |
| 0 | 32.19 | 53.78 | 61.11 | 53.78 | 32.19 | 0 |
| 0 | 19.26 | 32.19 | 36.57 | 32.19 | 19.26 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

(3) Parabolic equation with obstacles

We solve the above example in (2) with some obstacles[2], in the center of the domain, we have the results as shown below.

Example (3):_ Solve the equation $U_t = C^2 \nabla^2 U$ with the $U = 0$ on the boundary of the square of side $6h$, and in the center of the square, and the initial value of U in the other all points are given by $U = xy(6h - x)(6h - y) / h^4$.

| K=0 | | | | | | |
|-----|----|----|----|----|----|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 25 | 40 | 45 | 40 | 25 | 0 |
| 0 | 40 | 0 | 0 | 0 | 40 | 0 |
| 0 | 45 | 0 | 0 | 0 | 45 | 0 |
| 0 | 40 | 0 | 0 | 0 | 40 | 0 |
| 0 | 25 | 40 | 45 | 40 | 25 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| K=1 | | | | | | |
|-----|-------|-------|-------|-------|-------|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 21.78 | 35.78 | 40.44 | 35.78 | 21.78 | 0 |
| 0 | 35.78 | 0 | 0 | 0 | 35.78 | 0 |
| 0 | 40.44 | 0 | 0 | 0 | 40.44 | 0 |
| 0 | 35.78 | 0 | 0 | 0 | 35.78 | 0 |
| 0 | 21.78 | 35.78 | 40.44 | 35.78 | 21.78 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| K=2 | | | | | | |
|-----|-------|-------|-------|-------|-------|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 17.63 | 23.81 | 25.93 | 23.81 | 17.63 | 0 |
| 0 | 23.81 | 0 | 0 | 0 | 23.81 | 0 |
| 0 | 25.93 | 0 | 0 | 0 | 25.93 | 0 |
| 0 | 23.81 | 0 | 0 | 0 | 23.81 | 0 |
| 0 | 17.63 | 23.81 | 25.93 | 23.81 | 17.63 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| K=3 | | | | | | |
|-----|-------|-------|-------|-------|-------|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 13.13 | 16.08 | 16.81 | 16.08 | 13.13 | 0 |
| 0 | 16.08 | 0 | 0 | 0 | 16.08 | 0 |
| 0 | 16.81 | 0 | 0 | 0 | 16.81 | 0 |
| 0 | 16.08 | 0 | 0 | 0 | 16.08 | 0 |
| 0 | 13.13 | 16.08 | 16.81 | 16.08 | 13.13 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

(4) Poisson Equation

The sixth order finite difference method can be applied to the Poisson equation, the formula is given by the equation(20), we can use more term on the right hand side of the following equation:

$$h^2 \nabla^2 = \frac{1}{6}(2H + X) - \frac{1}{420}(13H^2 + 16HX + 6X^2) + \frac{1}{630}(2H^3 + H^2 + 4HX^2) - \dots(4.1)$$

Such that express the right hand side of (d1) by the function $f(x, y)$.

$$6h^2 u(x_i, y_j) = 6h^2 f(x_i, y_j) + \left(\frac{h^4}{2}\right) \nabla^2 f(x_i, y_j) + \left(\frac{h^6}{60}\right) \nabla^4 f(x_i, y_j) + \frac{h^6}{15} f_{xyy}(x_i, y_j) \dots(4.2)$$

Example (4.1)

Applied Poisson equation in a rectangular domain $0 \leq x \leq 1$, and $0 \leq y \leq 1$. Where the boundary condition and the exact solution is given by the equation $u(x, y) = \sin(\pi x) \sin(\pi y)$. Solution: we take multi step length to divided the square domain and we take the first term on the hand side of (4.2), obtained the results shown in table-1-. As well if we take two terms from the right hand side obtained the results shown in tabl-2-. By comparing the approximate solutions with the exact, we have the following results

Table-1-

| Step length h | Max absolute error using one term by 9- points |
|---------------|--|
| 0.25 | 1.07215197317066E-01 |
| 0.2 | 6.11825763467646E-02 |
| 0.1 | 1.657949043637963E-02 |
| 0.05 | 4.124499304353879E-03 |

Table-2-

| Step length h | Max absolute error using two terms by 9- points |
|---------------|---|
| 0.25 | 6.759250669847505E-03 |
| 0.2 | 2.450206780333142E-03 |
| 0.1 | 1.643228010298481E-04 |
| 0.05 | 1.019366904331243E-05 |

If we take more terms from the right hand side the solution is very significant as it is clear in the difference between two results in the two tables.

Remark: All the computations of the above results are in personal computer (Laptop) using FORTRAN language.

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الفروق المقسومة من الرتبة السادسة في بعدين مع أمثلة

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المستخلص

الفروق المقسومة بتسعة نقاط من الرتبة السادسة استعملت لحل معادلة لابلاس في متغيرين. كفاءة هذه الطريقة واضحة من خلال تطبيقها ومقارنتها مع الحل المضبوط. الحل الصريح لمعادلة القطع المكافئ إضافة إلى تطبيقها على نطاق يحتوي على مناطق معزولة، وكذلك يمكن تطبيقها على معادلة باسون مع إمكانية زيادة دقتها بإضافة عدد أكثر من الحدود من الطرف الأيمن. والنتائج العددية لهذه الطريقة مبينة في الأمثلة التي توضح ذلك.

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